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# An Optimized Algorithm to Determine The Eigenvalues From a Graphic Sequence by Constructing a Symmetric Matrix

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**Abstract.** Spectral graph theory is a popular topic in modern day applied mathematics. Determining the eigenvalues of a given graph gives us an in-depth idea about some interesting properties of the graph. A finite sequence of nonnegative integers is said to be graphical if there exists a finite simple graph, such that the degrees of its vertices corresponds to the terms of the sequence. Such a graph is often termed as a realization of the given degree sequence. In this paper we have proposed an algorithm that determines the realization of a given degree sequence by constructing the adjacency matrix from the given sequence. Further we utilize the adjacency matrix thus obtained to determine the eigenvalues of the graph.

**Keywords:** Spectral Graph Theory, Eigenvalues, Adjacency Matrix, Degree Sequence, Algorithm, Graphic Sequence

## 1. Introduction

Graph theory has been extensively used in various fields' viz. computer science, electrical engineering, civil engineering, management studies, operation research and many more. With the improvement of theories relating to graphs and linear algebra, spectral graph theory came into limelight. Algebraic graph theory is the branch of mathematics that deals with the study of graphs by using algebraic properties of associated matrices. Two common matrices that are widely used are the adjacency matrix and the Laplace matrix. A good definition of spectral graph theory that is due to Brouwer and Haemers [19], that defines spectral graph theory as the study of the relation between the graph properties and the spectrum of the adjacency matrix and the Laplace matrix.

One of the most common properties of a square matrix is its eigenvalue. The study of graph eigenvalues realizes increasingly rich connections with many other frontiers of mathematics. The spectrum of a graph when analysed with eigenvalues, yields a good resemblance with most of the major invariants of a graph, linking one extremal property to another. A good number of literatures exist that covers the important aspects of the spectral graph theory [17, 18, and 19].

Although it is quite easy to determine the adjacency matrix and hence its eigenvalues for a given graph, the process becomes quite complex when only a degree sequence is provided. In fact, we first need to verify whether the given sequence is graphic. Provided the sequence is graphic, we then try to draw the graph and finally prepare its adjacency matrix to determine the eigenvalues of the graph realized by the given degree sequence.

A finite sequence d: $d_1, d_2, d_3, \ldots, d_n$ of nonnegative integers is said to be graphical if there exists some finite simple graph G, having vertex set $V=\{v_1, v_2, v_3, \ldots, v_n\}$ such that each $v_i$ has degree $d_i$ $(1 \le i \le n)$. Renowned solutions were proposed independently by Havel [1] and Hakimi [2] in the mid 20th century. Although they provided separate proofs yet their work is jointly known as the Havel-Hakimi theorem.

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Two necessary conditions for a sequence to be graphical are: (1) di < n for each i and (2) $\sum_{i=1}^{n} d_i$ is even. However none of these are sufficient condition for the sequence to be graphic. With the advancement of discrete mathematics and linear algebra, various reputed sufficient conditions were discovered. Authors like Erdős and Gallai [3], Ryser [4], Berge [5], Fulkerson, Hofman and McAndrew [6], Bollobás [7], Grűnbaum [8] and Hässelbarth [9] independently proposed the sufficient condition for a degree sequence to be graphic. A good number of survey papers regarding the sufficiency criteria are also available. One popular work was due to Sierksma and Hoogeveen [10], where they have listed the seven well known characterizations of a degree sequence and their equivalence. Many authors came up with alternative proofs and extensions of the existing works. The works of Havel-Hakimi were further extended by Kleitman and Wang [14], and that of Erdős and Gallai by Eggleton [16] and Tripathi and Vijay [11]. Dahl and Flatberg [12] proposed a direct way of obtaining Tripathy and Vijay's result from a simple geometrical observation involving weak majorization. Tripathy and Tyagi [13] provided two elegant proofs of Havel-Hakimi and Erdős and Gallai.

In this paper we have developed an algorithm that can construct the adjacency matrix of a non-regular simple graph directly from the degree sequence. Once we obtain the required adjacency matrix, we then determine the eigenvalues of the graph. Thus we propose a methodology that can directly obtain the graph eigenvalues from the degree sequence of the graph.

In the following section we give a brief definition of the problem. Section 3 provides some of the existing criteria. In section 4 we present our proposed algorithm followed by a discussion of the results in section 5. We draw a brief conclusion in section 6.

## 2. Problem Definition

The proposed problem can be decomposed into two parts. The first part is to determine whether the degree sequence is graphic and to construct the corresponding adjacency matrix if the given degree sequence is graphic. The next part is to calculate the eigenvalues from the constructed matrix. We explain each in brief in the following two sub-sections.

### 2.1. Determination of Graphic Sequence

Let G=(V, E) be a finite simple graph with vertex set V and order n. Let each vertex $v_i$ has a degree $d_i$ where $1 \le i \le n$. Then the finite sequence d: $d_1, d_2, d_3, \ldots, d_n$ of nonnegative integers is called a degree sequence of the graph G. The problem statement can be formally stated as follows:

*Given a finite degree sequence d: $d_1, d_2, d_3, \ldots, d_n$ of nonnegative integers, whether there exists a graph G of order n with vertex set V such that each vertex $v_i$ has a degree $d_i$ where $1 \le i \le n$.*

### 2.2. Calculation the Eigenvalues

Given a square matrix *A*, the condition that characterizes an eigenvalue, λ, is the existence of a *nonzero* vector x such that $A x = \lambda x$; this equation can be rewritten as follows:

$Ax = \lambda x$

$Ax - \lambda x = 0$

$Ax - \lambda I x = 0$

$(A - \lambda I)x = 0$

This final form of the equation makes it clear that *x* is the solution of a square, homogeneous system. If *nonzero* solutions are desired, then the determinant of the coefficient matrix—which in this case is $A - \lambda I$—must be zero; if not, then the system possesses only the trivial solution x = 0. Since eigenvectors are, by definition, nonzero, in order for *x* to be an eigenvector of a matrix *A*, λ must be chosen so that

$det(A - \lambda I) = 0$

When the determinant of $A - \lambda I$ is written out, the resulting expression is a monic polynomial in λ. [A *monic* polynomial is one in which the coefficient of the leading (the highest-degree) term is 1.] It is called the characteristic polynomial of *A* and will be of degree *n* if *A* is $n \times n$. The zeros of the characteristic polynomial of *A*—that is, the solutions of the characteristic equation, $\det(A - \lambda I) = 0$—are the eigenvalues of *A*. Some of the popular method for finding the eigenvalues of a given square matrix can be found in [20].

## 3. Existing Criteria

A good number of criteria exist that are sufficient for a degree sequence to be graphic. Here we present some popular characterizations that are widely used. The details proof of most of these criteria can be found in [10]. The most popular criterion was proposed independently by Havel [1] and Hakimi [2] and is commonly referred to as the Havel-Hakimi theorem as stated below.

**Theorem Havel-Hakimi** (See [1] and [2]). *Let d: $d_1, d_2, d_3, \ldots, d_n$ be a finite non-decreasing sequence of nonnegative integers. Then the sequence is graphic if and only if the sequence $d_2$-1, $d_3$-1, ... , $d_{d_1+1}$ -1, $d_{d_1+2}$, ... , $d_n$ is graphic.*

Another well known criterion due to Erdős and Gallai [3] is stated below.

**Theorem Erdös and Gallai** (See [3]). *Let d: $d_1, d_2, d_3, \ldots, d_n$ be a finite non-decreasing sequence of nonnegative integers. Then the sequence is graphic if and only if $\sum_{i=1}^{n} d_i$ is even and the inequalities $\sum_{i=1}^{n} d_i \leq s(s-1) + \sum_{i=s+1}^{n} \min\{s, d_i\}$ holds for each s for $1 \leq s \leq n$.*

## 4. Proposed Algorithm

Here we propose an algorithm that determines whether the given degree sequence is graphical by constructing the adjacency matrix corresponding to the given degree sequence. The degree sequence is stored in a vector *degree[n]* of length n. The vector *Allocated[n]* of length n helps us to determine whether a given position in the adjacency matrix can be allocated 1, based on checking the number of positions allocated in the $j^{th}$ column of the $i^{th}$ row with respect to the degree of vertex *j*. The n *x* n vector *Adj[n][n]* is the adjacency matrix that we construct using the algorithm. Once the entire matrix is constructed, the algorithm then checks whether the resulted matrix is symmetric. The input to the algorithm is the given degree sequence in a non-increasing order and its output is the decision (i.e. graphic or non-graphic). The adjacency matrix *Adj[n][n]* constructed here is used to calculate the eigenvalues of the given graphic sequence.

ALGORITHM-CONSTRUCT-ADJACENCY(*d: $d_1, d_2, d_3, \ldots, d_n$*)

```
for i ← 1 to n
    Allocated[i] ← 0;
    degree[i] ← d_i;
r ← degree[0];
sum ← 0, flag ← 0, rflag ← 0;
for i ← 1 to n
    if degree[i] ≥ n
        return Non-Graphic;
    if degree[i] ≠ r
        rflag ← rflag + 1;
    sum ← sum + degree[i];
if sum mod 2 ≠ 0
    return Non-Graphic;
if sum mod 2 = 0 and rflag = 0
    return Graphic and Regular;
for i ← 1 to n
    k ← degree[i];
    for j ← 1 to n
        if k > 0
            if i = j
                Adj[i][j] ← 0
            else
```

```
            if Allocated[j] < degree[j]
                Adj[i][j] ← 1;
            Allocated[j] ← Allocated[j] + 1;
            k ← k – 1;
            else
            Adj[i][j] ← 0;
    if k > 0
        return Non-Graphic;
if Adj[i][j] symmetric
    calculate eigenvalues;
    return Graphic;
else
return Non-Graphic;
```

## 5. Results and Discussion

The following table 1 depicts the results of the proposed algorithm versus the output of well known Havel-Hakimi theorem.

Table 1: Results of the proposed algorithm with corresponding eigenvalues.

| Degree Sequence | Havel-Hakimi output | Proposed Algorithm Output | Eigen values |
|---|---|---|---|
| $d_1$: 5,4,3,3,2,2,2,1,1,1 | Graphic | Graphic | -1.9811, -1.4585, -1.0000, -1.0000, -1.0000, 0.1758, 0.3966, 1.0000, 1.4549, 3.4122 |
| $d_2$: 7,7,4,3,3,3,2,1 | Non-Graphic | Non-Graphic | Not Applicable |
| $d_3$: 5,3,3,3,3,2,2,2,1 | Graphic | Graphic | -1.8527, -1.4915, -1.0000, -1.0000, -1.0000, 0.1053, 1.1771, 1.7773, 3.2844 |
| $d_4$: 6,3,3,3,3,2,2,2,2,1,1 | Graphic | Graphic | -1.8598, -1.6180, -1.2912, -1.0000, -1.0000, -0.6180, -0.0000, 0.6180, 1.6180, 1.6791, 3.4719 |
| $d_5$: 7,6,5,4,4,3,2,1 | Graphic | Graphic | -2.2643, -1.4119, -1.2434, -1.0000, 0.2319, 0.3341, 0.6973, 4.6564 |

In the above table, all the degree sequences except $d_2$ is graphic. Whenever the algorithm gets a graphic sequence, it calculates the eigenvalues prints it. From the table it can be easily verified that the algorithm rightly identifies the graphic sequences and prints their eigenvalues. Since $d_2$ is non-graphic, the algorithm cannot construct the adjacency matrix and hence its eigenvalues cannot be determined.

## 6. Conclusion

The algorithm presented in this paper works well for the degree sequence of all types of graphs. The algorithm generates the results by applying some basic criteria and constructing the adjacency matrix. The adjacency matrix generated in this process has been used to determine the eigenvalues of the graph. Further this matrix can be utilized to extract some useful information from the degree sequence like the eigen space of a given graphic sequence, connectivity of the graph etc. While the Havel-Hakimi is a recursive greedy approach, the proposed algorithm is an iterative one. It can also be applied to various branches of combinatorial analysis and other renowned problems such as communication networks, stereochemistry, etc. Thus we can conclude that the proposed algorithm is a good approach to determine whether a given degree sequence is graphic.

## 7. References


[1] Havel V. A remark on the existence of finite graphs (Czech.). Ćasopis Pěst. Mat. 1955;80: 477-480.

[2] Hakimi S L. On the realizability of a set of integers as degrees of the vertices of a graph. SIAM J. Appl. Math. 1962;10: 496-506.

[3] Erdős P, Gallai T. Graphs with prescribed degree of vertices (Hungarian). Mat. Lapok. 1960;11: 264-274,.

[4] Ryser HJ. Combinatorial properties of matrices of zeros and ones. Can. J. Math. 1957;9:371-377.

[5] Berge C. Graphs and Hypergraphs. American Elsevier, New York, Amsterdam: North Holland; 1973.

[6] Fulkerson DR, Hofman AJ, McAndrew MH. Some properties of graphs with multiple edges. Can. J. Math. 1965;17:166-177.

[7] Bollobås B. Extremal Graph Theory. New York: Acedemic Press; 1978.

[8] Grű̆nbaum B. Graphs and complexes. Report of the University of Washington, Seattle, Math., 572B, 1969 (private communication).

[9] Hӓsselbarth W. Die Verzweightheit von Graphen. Match 1984;16:3-17.

[10] Sierksma G, Hoogeveen H. Seven criteria for integer sequences being graphic. Journal of the Graph Theory 1991;15 no. 2:223-231.

[11] Tripathi A, Vijay S. A note on theorem on Erdős and Gallai. Discrete Mathematics. 2003;265:417-420.

[12] Dahl G, Flatberg T. A remark concerning graphical sequences. Discrete Mathematics. 2005;304:62-64.

[13] Tripathi A, Taygi H. A simple criterion on degree sequences of graphs. Discrete Applied Mathematics. 2008;156:3513-3517.

[14] Kleitman DJ, Wang DL. Algorithms for constructing graphs and digraphs with given valences and factors. Discrete Math 1973;6:79-88.

[15] Bondy JA, Murty USR. Graph Theory with Applications. American Elsevier, New York, Amsterdam: North Holland; 1976.

[16] Eggleton RB. Graphic sequences and graphic polynomials: A report. Infinite and Finite Sets 1973;1:385-392.

[17] Biggs N. Algebraic Graph Theory. Cambridge: Cambridge University Press;1993.

[18] Cvetković DM, Doob M, Sachs H. Spectra of graphs, theory and application. Academic, New York; 1980.

[19] Brouwer AE, Haemers WH. Spectra of graphs. Springer; 2011.

[20] Rosen KH. Handbook of Linear Algebra. CRC Press; 2007.