Diversification and Robust Measures of Tail Risk in Mutual Funds

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Abstract. This paper examines the efficiency of market standard portfolios (Markowitz and Sharpe) through the effects of diversification and the tail risk of mutual funds portfolio by applying higher moments and various robust measures of tail weight. We analyze the monthly returns of U.S. Equity mutual funds from October 2009, to September 2011. By carrying out extensive simulations using bootstrap methods, we find that market standard portfolios, Markowitz and Sharpe ratio optimization portfolio, are exposed to higher tail risk than Naïve portfolio, *1/N* equally weighted portfolio. We also find that the diversification effect holds in Naïve portfolios, while Markowitz and Sharpe ratio optimization portfolios show little evidence thereof. In conclusion, the optimized portfolio based on the mean-variance analysis leads to extremely non-optimal asset allocation in mutual funds portfolio and this phenomenon is more evident in case when tail risk is measured applying the robust measures.

Keywords: Diversification, tail risk, robust higher moment, robust tail weight, mutual funds.

1. Introduction

This paper examines the effects of diversification, defined as the number of underlying funds in a fund portfolio, and tail risk in mutual funds portfolios. There are several methods for modelling tail risk: the co-movement approach proposed by Rubinstein (1973), copula function and extreme value theory. As Brown and Spitzer (2006) point out, such methods are difficult to implement with mutual fund returns since they are reported monthly and generally have short time series. Brown and Spitzer (2006) focus on the cross-sectional approach and argue that, through aggregation, the extra dependence would be indistinguishable from noise in individual series. Advocating their viewpoint, this paper applies a cross-sectional aggregation of various tail risk measures with respect to the number of funds in a portfolio to investigate the efficiency of market standard (Markowitz and Sharpe) portfolios, focused on diversification and tail risk. In particular, we highlight the tail risk of mutual funds using various tail-related measures from classical skewness to robust tail measures.

The stylized facts of financial returns are summarized as financial returns exhibit negative skewness, excess kurtosis, and clustering in volatility dynamics. Many alternatives have been proposed to capture these phenomena, including fat tail distributions such as generalized hyperbolic distributions and GARCH-type models for time-varying volatility. One alternative is to explicitly combine higher moments, such as skewness and kurtosis, for the asset pricing, risk measurement and asset allocation. For example, Harvey and Siddiqui (1999) propose an asset pricing model that incorporates skewness, and Jondeau and Rockinger (2006) study optimal asset allocation using higher moments. Skewness can be interpreted as a measure of the asymmetry of the probability density function. The coefficient of kurtosis is generally regarded as a measure of tail heaviness of a distribution relative to that of the normal distribution; however, it also measures the peakedness of a distribution, and it is thus confusing to employ kurtosis to measure tail risk. A common problem with such measures is their extreme sensitivity to outliers, thus we need a robust measure for the tail of a distribution. To achieve this goal, several measures of robust skewness and left (right) tail weight for univariate continuous distributions are proposed. This paper introduces robust measures of skewness and robust tail weight measures to examine portfolio tail risk with robustness.

Recently, Brown, Gregoriou, and Pascalau (2012) studied the relations between diversification, defined as the number of funds in a fund of hedge funds (FoHF), and tail risk using FoHF monthly returns. They consider a simple measure of tail risk based on classical skewness and kurtosis. They conclude that the variance-reducing effects of diversification diminish once an FoHF holds more than 20 underlying hedge

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funds. They argue that this excess diversification actually increases their left-tail risk exposure, expressed as negative skewness. The size of assets under management in mutual funds sector is larger than any other delegated asset classes, therefore it is important to investigate tail risk behaviours of mutual fund portfolios with consistent and robust manners.

To the best of our knowledge, this study is the first to measure portfolio tail risk based on various robust tail measures such as the quantile weight and medcouple. We investigate the tail risk behaviours of fund portfolios with respect to the number of underlying funds, which is a simple and intuitive indicator of diversification. Empirical results show that portfolio tail risk measured by classical skewness and robust tail measures reveal contradictory results. By construction, classical skewness is sensitive to data outliers and reflects the right and left tails at the same time, which leads to misunderstandings in the conceptual sense of risk. Based on empirical results, we conclude that robust measures of skewness are suitable for measuring tail risk. This paper considers five types of portfolios: (1) equally weighted, (2) maximizing returns without risk constraints, (3) maximizing returns with risk constraints, (4) maximizing the Sharpe ratio, and (5) minimizing risk with return constraints. The first two portfolios serve as a benchmark in our study, and the last three portfolios are regarded as market standards. Empirical results show that the equally weighted portfolio outperforms the optimized portfolios, based on the trade-off relation between return and risk. We summarized our findings as follows: The optimized portfolios based on mean-variance analysis are extremely inefficient since these methods are exposed to too much tail risk. In addition, the diversification effect of equally weighted portfolios is evident based on variance and kurtosis, whereas the mean and skewness remain almost at the same level, regardless of increases in the number of funds in portfolios. However, optimized portfolios decrease variance, however increase left-tail risk and kurtosis as the number of funds increases.

The rest of this paper is organized as follows. Section 2 briefly introduces robust tail risk measures, focusing on robust skewness and tail weight measures. Section 3 discusses the data and the results of our empirical analyses. Section 4 concludes.

2. Robust Tail Measures

2.1. Robust Skewness

Let r_t, for $t = 1,...,T$, be the return series of interest and assume the independent and identically distributed with a cumulative distribution function F. If up to the fourth moments exist, the skewness for r_t is given by

$$SK_1 = E\left(\frac{r_t - \mu}{\sigma}\right)^3, \tag{1}$$

where $\mu = E(r_t)$ and $\sigma^2 = E(r_t - \mu)^2$. Due to the higher-power terms in equation (1), it is evident that the measures of skewness and kurtosis are influenced by one or more outliers in the data of interest.

Bowley (1920) proposes an alternative measure of skewness based on quantiles, defined as

$$SK_2 = \frac{Q(0.75) + Q(0.25) - 2Q(0.5)}{Q(0.75) - Q(0.25)}, \tag{2}$$

where Q denotes the quantile function of the distribution F such that $Q(\alpha) = F^{-1}(\alpha)$. The value of SK_2 is zero for any symmetric distribution and its range is (-1, 1). The value of SK_2 close to one indicates extreme right skewness, and that close to -1 indicates extreme left skewness.

Hinkley (1975) proposes a generalization of Bowley's coefficient of skewness,

$$SK_3(\alpha) = \frac{Q(1-\alpha) + Q(\alpha) - 2Q(0.5)}{Q(1-\alpha) - Q(\alpha)}, \tag{3}$$

for any α between zero and 0.5. To remove dependence on the value of α, Groeneveld and Meeden (1984) develop Hinkley's skewness to integrate out α, leading to the expression

$$SK_3 = \frac{\mu - Q_2}{E|y_t - Q_2|}. \quad (4)$$

Similar to Bowley's coefficient of skewness, SK_3 is also zero for any symmetric distributions and is bounded by -1 and one.

The common characteristic of the skewness in formula (1) to (4) three different measures of is that they consider the left and right tails simultaneously. If we focus only on the tail of one side, these measures are insufficient. To overcome this problem, we introduces the concepts of Left Quantile Weight (LQW) and the Left MedCouple (LMC).

2.2. Robust Measures of Tail Weight

Following the study of Brys, Hubert, and Struyf (2006), we introduce left and right tail measures as measures of skewness that are applied to the half of the probability mass lying on the left and right sides, respectively, of the median of F. The LQW is applied to the half of the probability mass lying to the left side of the median of F. Formally, the LQW is defined as

$$LQW_F(p) = -\frac{\left(Q\left(\frac{1-p}{2}\right) - Q\left(\frac{p}{2}\right)\right) - 2Q(0.25)}{Q\left(\frac{1-p}{2}\right) - Q\left(\frac{p}{2}\right)}, \quad (6)$$

where $0 < p < 0.5$. To retain a reasonable amount of robustness, we set p as follows, $LQW_F(0.125)$ and $LQW_F(0.25)$. 0.25 is closely related to the Bowley's skewness in formula (2) and 0.125 is related to octile skewness.

The medcouple is defined as a scaled median difference of the left and right halves of the distribution and is thus not based on the third moment, as classical skewness is. The medcouple is

$$MC(F) = \underset{x_1 \le m_F \le x_2}{med} h(x_1, x_2), \quad (7)$$

where x_1 and x_2 sampled from F and m_F denotes median of the distribution F. The kernel function given by

$$h(x_i, x_j) = \frac{(x_j - m_F) - (m_F - x_i)}{x_j - x_i}, \quad (8)$$

We can easily apply the medcouple to just one side of the distribution, leading to the LMC. Formally, LMC is defined as

$$LMC_F = -MC(x < m_F). \quad (9)$$

The quantile weight and medcouple can both be applied to symmetric as well as asymmetric distributions. Their interpretation is clear and they are robust to outlying values. Both the quantile weight and medcouple measures depend only on quantiles and consequently can be computed for any distribution, even without finite moments.

3. Empirical Results

Empirical study uses the monthly returns of U.S. Equity mutual funds obtained from the mutual fund database of the Center for Research in Security Prices (CRSP). Our sample ranges from October, 2009 to September, 2011. Based on the final calculated return date, September 30, 2011, and Lipper asset classifications, there are 19,576 "U.S. Equity" funds out of 27,999 funds.

To conduct empirical analysis, we apply a random selection with replacement, known as the bootstrap method, to construct hypothetical fund portfolios from this database. We repeat this exercise to construct hypothetical portfolios with five to 50 underlying funds and repeat this experiment 25,000 times. We exclude portfolios whose excess returns are negative during random sampling, since such portfolios are never actually chosen in reality. In this way we can compute such risk measures as variance and skewness for each hypothetical fund.

We consider five types of portfolios: (1) equally weighted (*Naïve*), (2) maximizing returns without risk constraints (*MaxRet*), (3) maximizing returns with risk constraints (*Markowitz 2*), (4) maximizing the Sharpe ratio (*Sharpe*), and (5) minimizing risk with return constraints(*Markowitz 1*). The first two portfolios serve as a benchmark in our study, and the last three portfolios are regarded as market standards.

Fig. 1 presents the average of variance and skewness of hypothetical fund portfolios with varying the number of underlying funds, by applying cross-sectional aggregation. As anticipated, *MaxRet* shows the highest variance and *Markowitz 1* shows the lowest risk, followed by the *Sharpe* portfolios. Based only on the average skewness, the *Markowitz 1* portfolios have the worst performance, followed by the Sharpe ratio portfolios. Additionally, these two optimized portfolios shows the increasing function of left-tail risk as the number of funds increase. This results show that optimized portfolios based on mean-variance analysis expose unexpected left-tail risk, thus lead inefficient asset allocation. The skewness of Naïve portfolios remains almost the same with respect to an increase in the number of underlying funds. The *MaxRet* portfolios exhibit the highest skewness, which means *MaxRet* portfolios are the most preferred, based on classical skewness.

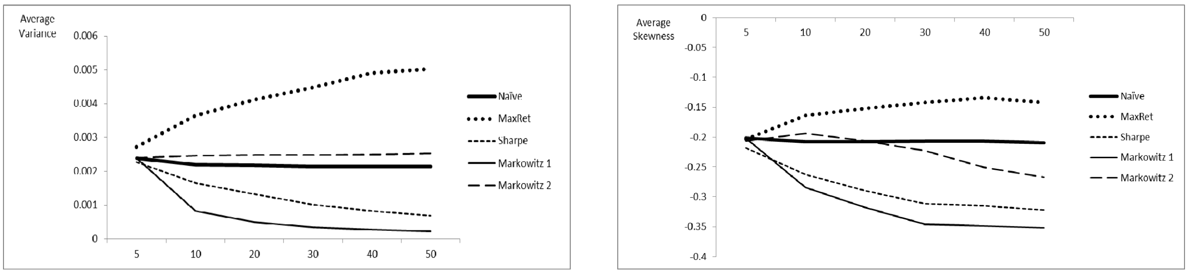

Fig. 1: The average of variance and skewness of mutual funds portfolios, applying cross-sectional aggregation.

Fig. 2 contains two robust measures of skewness (SK_2 and SK_3) and provides interesting and remarkable findings. As mentioned before, based only on classical skewness, *MaxRet* shows the best performance and the *Markowitz 1* portfolio has the worst performance, followed by the Sharpe portfolio. However, considering robust skewness, *MaxRet* portfolios are riskier than any other portfolios, which coincide with our initial expectation; *MaxRet* is the riskiest portfolio since it does not consider any risk. Therefore, we can conclude that robust measures of skewness are conceptually suitable for measuring tail risk.

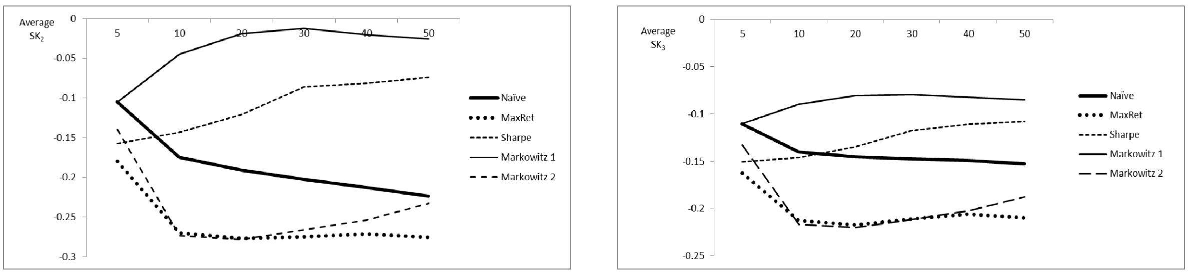

Fig. 2: The average of robust skewness (SK_2 and SK_3) of mutual funds portfolios, applying cross-sectional aggregation.

We now discuss the measure of the left tail weight using two measures: LQW(0.25) and the LMC. Fig. 3 shows very similar patterns of Fig. 2, as the number of underlying funds increases. We obtain a positive (negative) LQW measure if the lower quantile is skewed to the left (right). The *Markowitz 1, Naïve*, and *Sharpe* portfolios have positive LQW measures, regardless of the number of underlying funds. The *Markowitz 1* and *Sharpe* portfolios show positive LQW and LMC values as the number of underlying funds increases, which means these portfolios may generate a distribution of portfolio returns skewed to the left. Thus we conclude that these two portfolios are greatly exposed to tail risk.

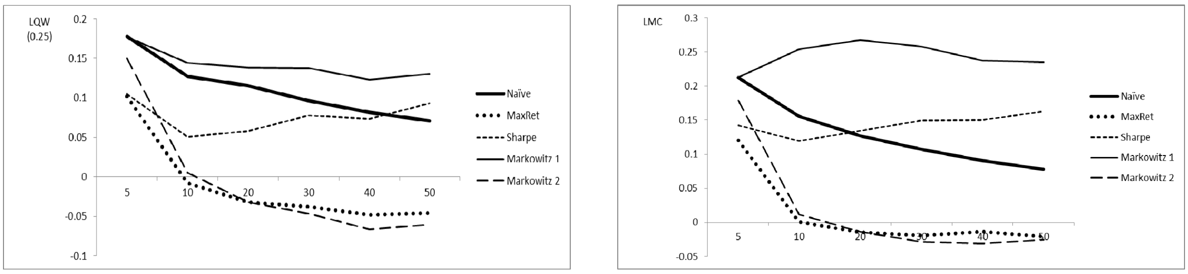

Fig. 3: The average of LQW(0.25) and the LMC of mutual funds portfolios.

4. Conclusion

This paper examines the effects of diversification and tail risk of U.S. mutual funds by adopting higher moments and various robust measures of tail risk. This paper introduces several tail weight measures based on robust measures of skewness. Carrying out extensive simulations, we compare conventional higher-moment measures with robust measures to capture the tail risk of given portfolios. Our empirical results show that market standard portfolios based on second moments are exposed to higher tail risk than naïve portfolios. Based on our results, portfolio optimization based on second moments provides extremely undesirable asset allocations when measuring tail risk using robust tail measures. We investigate the tail risk behaviours of fund portfolios with respect to the number of underlying funds, which is interpreted as the diversification effect. We find that the diversification effect holds in naïve portfolios, while Markowitz and Sharpe ratio optimization portfolios show little evidence thereof.

5. References

[1] A. Bowley. *Element of Statistics.* Scribner, New York, 1920.

[2] S. Brown, G. N. Gregoriou, and R. Pascalau. Diversification in funds of hedge funds: Is it possible to overdiversify? *Review of Asset Pricing Studies* (forthcoming), 2012.

[3] S. Brown, and J. F. Spitzer. Caught by the tail: Tail risk neutrality and hedge fund returns. Working paper, New York University Stern School of Business. 2006.

[4] G. Brys, M. Hubert, and A. Struyf. Robust measures of tail weight. *Computational Statistics and Data Analysis.* 2006, **50**: 733–759.

[5] R. A. Groeneveld, and G. Meeden. Measuring skewness and kurtosis. Statistician. 1984, **33**: 391–399.

[6] C. R. Harvey, and A. Siddiqui. Autoregressive conditional skewness. *Journal of Finance and Quantitative Analysis*. 1999, **34**: 465–487.

[7] D. Hinkley. On power transformation to symmetry. *Biometrika*. 1975, **62**: 101–111.

[8] E. Jondeau, and M. Rockinger. Optimal portfolio allocation under higher moments. *Journal of the European Financial Management Association*. 2006, 12, 29–55.

[9] D. Ruppert. What is kurtosis? An influence function approach. American Statistician, 1987, **41**: 1–5.

[10] M. E. Rubinstein. The fundamental theorem of parameter preference security valuation. *Journal of Financial and Quantitative Analysis*. 1973, **8(1)**: 61–69.