Measuring Risk Dependencies Due to Two Natural Disasters by Bivariate Copula

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Abstract. The bivariate copula potentiality for measuring risk dependencies due to occurrence of two natural disasters is presented. Some bivariate copula essentials are described. Several copula dependence measures (concordance, Kendall's tau, Spearman's rho, positively quadrant dependent, tail dependence) are considered. The results about the measuring risk dependencies can support the stakeholders to take more informed decisions regarding the efficient allocation of the available funding for the improvement of risk management with respect to natural disasters. A concept for implementing the bivariate copula models as a part of a Web integrated information system for risk management of natural disasters is outlined.

Keywords: bivariate copula, risk dependencies, Kendall's tau, Spearman's rho, natural disasters

1. Introduction

Today it is recorded an increase in negative severities of natural disasters on the life quality compared to previous years [1]. Billions of dollars cost annual losses resulting from floods, hurricanes, earthquakes, tornadoes, landslides etc. Unfortunately, natural disasters are impossible to avoid and infrastructure systems cannot be made totally invulnerable. The only feasible strategies for risk management and consequences reduction can be designed [2, 3].

For these reasons, the numerous scientific and applied investigations are conducted on separate natural disaster and their consequences for the people's health and property, the environment, cultural and material assets. The scientific activities are concerned to the development and application of modern methods and tools for analysis and estimation of risk events; development of models for risk control in emergency situations; development and maintenance of dedicated databases and information systems.

It is important to note that mostly the interdependence of natural disasters and their joint impact on society and infrastructure are not sufficiently taken into account in scientific study. Therefore it is necessary to propound varied mathematical methods for measuring risk dependencies due to natural disasters.

One method of modelling risk dependencies which has become very popular recently is the copula [4, 5]. The word copula is a Latin noun which means 'a link, tie or bond', and was first employed in a mathematical or statistical sense by Abe Sklar. Mathematically, a copula is a function which allows us to combine univariate distributions to obtain a joint distribution with a particular dependence structure [6]. The copulas provide full information on the dependency structure between risks. The concept of copulas is based on separating the joint marginal distribution function into a part that describes the dependence structure and multiple parts that describe the marginal distribution functions [7-9].

The purpose of the paper is to present the bivariate copula potentiality for measuring risk dependencies due to occurrence of two natural disasters. The described bivariate copula elements are envisaged to be implemented as a part of a Web integrated information system for risk management of natural disasters.

2. Bivariate Copula Essentials

The 2-dimensional copula links the bivariate cumulative distribution function to its one-dimensional marginals cumulative distribution function and therefore, it carries the dependence structure between these

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marginals [7]. The bivariate copula (in particular, 2-dimensional copula) is defined by Sklar as a bivariate distribution function with margins that are uniformly distributed on $[0,1]$:

$$C(x,y)=P(X\le x, Y\le y), \tag{1}$$

where $C(x,y)$ is the copula; $(X,Y)^T$ with $X\in U(0,1)$ and $Y\in U(0,1)$ is a vector of random variables and $(x,y)^T\in[0,1]^2$ are realizations of $(X,Y)^T$.

The 2-dimensional copula $C(x,y)$ is in fact a joint distribution function on $[0,1]^2$ with standard uniform marginals, where X and Y are uniformly distributed on $[0,1]$.

The bivariate copula $C(x,y)$ is a mapping from the unit square $[0,1]^2$ to the unit interval $[0,1]$. It is increasing in each component and satisfies the following conditions [:

1. $C(1,y)=y$ and $C(x,1)=x$ for $0\le x,y\le 1$;
2. $C(x,0)=0=C(0,y)$ for $0\le x,y\le 1$; (2)
3. for any $0\le a_1\le a_2\le 1$ and $0\le b_1\le b_2\le 1$, $C(a_2,b_2)-C(a_2,b_1)-C(a_1,b_2)+C(a_1,b_1)\ge 0$

The first and second conditions above imply that the marginal distribution of each component of the copula is uniform. The third condition is called the rectangle inequality. It ensures that $P(a_1\le X\le a_2,\ b_1\le Y\le b_2)$ is nonnegative.

A central result to the copula theory is Sklar's theorem (1959). This theorem states the representation of the joint distribution function using a copula. It also shows how a joint distribution can be created via a copula [6, 9].

Specifically, for the bivariate case, Sklar's theorem is:

Given the joint and marginal distribution functions of X and Y, there exists a unique copula $C(.,.)$, such that

$$F_{XY}(x,y)=C(F_X(x),F_Y(y)). \tag{3}$$

Conversely, if $C(.,.)$ is a copula, and $F_X(x)$ and $F_Y(y)$ are marginal (univariate) distribution functions of X and Y, respectively, then $C(F_X(x),F_Y(y))$ is a bivariate distribution function with marginal distribution functions $F_X(x)$ and $F_Y(y)$.

If the inverse functions $F_X^{-1}(.)$ and $F_Y^{-1}(.)$ exist, the copula satisfying (3) is given by:

$$C(u,v)=F_{XY}\left(F_X^{-1}(u),F_Y^{-1}(v)\right) \tag{4}$$

The second part of Sklar's theorem enables to construct a bivariate distribution with given marginals. With a well-defined copula satisfying definition (2), $C(F_X(x),F_Y(y))$ establishes a bivariate distribution with the known marginals. This can be described as a bottom-up approach in creating a bivariate distribution.

The maximum and minimum of a copula are established by the theorem, called the Fréchet bounds. The following bounds is applied to any bivariate copula:

$$\max\{0,u+v-1\}\le C(u,v)\le \min\{u,v\}. \tag{5}$$

The likelihood function of a bivariate distribution created by a copula can be computed using the following theorem:

Let X and Y be two continuous distributions with probability density function $F_X(x)$ and $F_Y(y)$, respectively. If the joint distribution function of X and Y is given by (3), their joint probability density function can be written as

$$f_{XY}(x,y)=f_X(x).f_Y(y).c(F_X(x),F_Y(y)). \tag{6}$$

where

$$c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}. \tag{7}$$

is called the copula density.

From (6)-(7), it can be concluded that the log-likelihood of a bivariate random variable with distribution function (3) is

$$\log(f_{XY}(x,y)) = \log(f_X(x)) + \log(f_Y(y)) + \log(c(F_X(x), F_Y(y))). \tag{8}$$

which is the log-likelihood of two independent observations of X and Y, plus a term which measures the dependence.

Therefore it can be derived a bivariate distribution function out of specified marginal distributions and a copula that contains information about the dependence structure between the single variables. Also the opposite holds: A copula can be determined out of the inverse of the marginal distributions and the bivariate distribution function [8].

3. Dependence Measures of a Copula

3.1. Concordance

Let (x_i, y_i) and (x_j, y_j) be two observations of a pair of random variables (X,Y).

It is said that (x_i, y_i) and (x_j, y_j) are concordant if $(x_i - x_j)(y_i - y_j) > 0$. That is to say that (x_i, y_i) and (x_j, y_j) are concordant if $x_i < x_j$ and $y_i < y_j$, or if $x_i > x_j$ and $y_i > y_j$.

Conversely, (x_i, y_i) and (x_j, y_j) are discordant if $(x_i - x_j)(y_i - y_j) < 0$.

It is necessary to point that concordance describes a pair of random variables in which, large values tend to be associated with large values and small values tend to be associated with small values.

3.2. Kendall's Tau

Kendall's τ (tau) is a measure of association between two random variables. It is defined in terms of concordance.

Let (x_i, y_i), $1 \le i \le n$ be a sample of n observations from (X,Y), a pair of continuous random variables. Let c denotes the number of concordant pairs of observations (x_i, y_i) and (x_j, y_j) with $(i,j) \in \{1,...,n\}^2$. Let d denotes the number of discordant pairs. Kendall's τ is defined as

$$\tau = \frac{c-d}{c+d} = \frac{c-d}{C_n^2},$$

where $C_n^2 = \frac{n!}{2!(n-2)!}$ is the binomial coefficient equals to the number of pairs.

Kendall's τ measure of a vector (X,Y) of random variables with joint distribution function (3) can be defined as the difference between the probabilities of concordance and discordance for two independent pairs (X_1, Y_1) and (X_2, Y_2) that is chosen randomly from the sample:

$$\tau_{XY} = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0).$$

These probabilities can be evaluated by integrating over the distribution of any pair (X_i, Y_i) of continuous random variables.

Hence, in terms of copulas, Kendall's τ can be expressed as follows:

$$\tau_C = 4 \int_0^1 \int_0^1 C(u,v) dC(u,v) - 1,$$

where C is the bivariate copula associated to (X,Y).

3.3. Spearman's Rho

Spearman's ρ (rho) is non-parametric correlation coefficient. It is also based on concordance and discordance. Let (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3) be three independent random variables with a common joint distribution (3).

Spearman's ρ is defined to be proportional to the probability of concordance minus the probability of discordance for the two vectors (X_1, Y_1) and (X_2, Y_3), that have the same margins but one has distribution function (3), while the components of the other are independent:

$$\rho_{XY} = 3(P((X_1 - X_2)(Y_1 - Y_3) > 0) - P((X_1 - X_2)(Y_1 - Y_3) < 0)).$$

In terms of the copula C associated to pair (X, Y), Spearman's ρ can be rewritten as:

$$\rho_C = 12\int_0^1\int_0^1 (C(u,v) - uv)dudv \qquad \text{or} \qquad \rho_C = 12\int_0^1\int_0^1 C(u,v)dudv - 3.$$

3.4. Positively Quadrant Dependent

The random variables X and Y with joint distribution function (3) are positively quadrant dependent (PQD) if for all (x, y) in R^2

$$P(X \le x, Y \le y) \ge P(X \le x)P(Y \le y),$$

or equivalently,

$$P(X > x, Y > y) \ge P(X > x)P(Y > y).$$

It can see that X and Y are positively quadrant dependent implies that the probability that they are simultaneously small (respectively, large) is at least as great as it would be if they were independent.

It can be said that X and Y are positively quadrant dependent if

$F_{XY}(x, y) - F_X(x), F_Y(y) \ge 0$ for all $x, y \in$ R.

In term of copula, this property can be rewritten as

$C(u, v) \ge uv$ for all $u, v \in [0,1]$.

3.5. Tail Dependence

Tail dependence coefficients are designed to capture the dependence between the marginals in the upper-right quadrant and in the lower-left quadrant of $[0,1]^2$.

The upper tail dependence parameter λ_U for a vector (X, Y) of random variables with joint distribution function (3) is defined as:

$$\lambda_U = \lim_{t \to 1^-} P\left(Y > F_Y^{-1}(t) \middle| X > F_X(t)\right).$$

It can be shown that

$$\lambda_U = 2 - \lim_{t \to 1^-} \frac{1 - C(t,t)}{1 - t}.$$

Analogously, it can be defined the lower tail dependence parameter λ_L with $t \to 0^+$.

If $\lambda_U = 0$ then it can be concluded that the copula C has no upper tail dependence. This means that if it is gone far enough into the upper tail of the joint distribution, then extreme events is appeared to occur independently.

The bivariate copula model is described only theoretically. Further research is needed particularly regarding its implementation for measuring risk dependencies due to occurrence of two natural disasters.

4. Conclusions

The bivariate copula essentials are presented. This copula is proposed to be used for measuring risk dependencies due to occurrence of two natural disasters. Several copula dependence measures (concordance, Kendall's tau, Spearman's rho, positively quadrant dependent, tail dependence) are considered. A concept for implementing the bivariate copula models as a part of a Web integrated information system for risk management of natural disasters is outlined.

The results about the measuring risk dependencies can support the stakeholders to take more informed decisions regarding the efficient allocation of the available funding for the improvement of risk management with respect to natural disasters.

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