2010 International Conference on E-business, Management and Economics
IPEDR vol.3 (2011) © (2011) IACSIT Press, Hong Kong

Stock Returns Declustering Under Time Dependent Hölder Exponent

Sergio Bianchi
Department of Economics and L.I.S.A.
University of Cassino
ITALY
e-mail: sbianchi@eco.unicas.it

Alexandre Pantanella
Ph.D. School of Economics and L.I.S.A.
University of Cassino
ITALY
e-mail: a.pantanella@eco.unicas.it

***Abstract*—Two of the most important stylized facts well-known in finance relate to the non-Gaussian distribution and to the volatility clustering of stock returns. In this paper, we show that a new class of stochastic processes – called Multifractional Processes with Random Exponent (MPRE) – can capture in a very parsimonious way both these "anomalies". Furthermore, we provide evidence that the sole knowledge of functional parameter characterizing the MPRE allows to calculate residuals that perform much better than those obtained by other discrete models such as the GARCH family.**

Keywords-Multifractional Processes with Random Exponents; Declustering; Residuals

I. INTRODUCTION

"[...] *As you well know, the biggest problem we now have with the whole evolution of risk is the fat-tail problem, which is really creating very large conceptual difficulties. Because, as we all know, the assumption of normality enables us to drop off the huge amount of complexity in our equations... Because once you start putting in non-normality assumptions, which is unfortunately what characterizes the real world, then these issues become extremely difficult.*"

Alan Greenspan (1997)

It is now well known that the Brownian-based standard financial theory is ineffective in matching most of the features that characterize actual data. "Anomalies" such as the highly non Gaussian marginal distributions, the non-autocorrelated log variations and the highly autocorrelated volatility process or the volatility bursts are all well documented in the literature and claim for a rethink of the very bases of standard models.

In order to address the gap between theory and practice, many models have been proposed, by relaxing the assumption of independence or that of identical distribution of the price variations. Just to quote the main research lines of the last quarter of century it suffices to think to the ARCH-like models in discrete time or to the stochastic processes with stable innovations, in continuous time.

In this paper we focus on a new class of stochastic processes, introduced in 2005 by Ayache and Taqqu [2] without any specific concern to financial time series: the *Multifractional Processes with Random Exponent* (MPRE). This family extends the *multifractional Brownian motion* (mBm) [3], [4] which in its turn generalised the very well-known *fractional Brownian motion* (fBm), by allowing the Hurst parameter to change in a deterministic way. Basically, the MPRE differs from the mBm in that it allows the functional Hurst (Hölder) parameter to evolve randomly. Whereas the mBm is generally disregarded in finance, mainly because its nonstationarity hampers the inference of global probabilistic properties, the MPRE is decidedly attractive because it has been proved to have stationary increments under a mild technical condition. This is the very first step needed to grasp how to conciliate the no-arbitrage principle with the stochastic kernel used to define the MPRE. From a practitioner viewpoint, many reasons justify the use of the MPRE as a model of the financial dynamics; detailed discussions can be found in [7], [8], [9], [10] and [11]. Here we restrict ourselves to mention the capability the process has to (a) provide a rationale for the trading mechanism, and (b) replicate the stylized facts observed in finance: fat tails and peaked centre of the distribution of returns; lack of dependence in returns, significant positive autocorrelation of the volatility process; volatility scaling (for a range of time intervals); heterogeneity (ubiquitous volatility clustering); non-linearity (see [14] and [15] for a survey).

The paper is organized as follows: Section II introduces the model and provides a simple proof of its distributional properties (high peaks and fat tails); in Section III we discuss an estimator of the functional parameter of the MPRE and in Section IV we perform an analysis on four main stock indices. Section V concludes.

II. THE MODEL

The best way to introduce the *Multifractional Processes with Random Exponent* (MPRE) is perhaps to start from the *fractional Brownian motion* (fBm), discussed in 1940 by Kolmogorov and revived in 1968 by Mandelbrot and Van Ness [5]. The fBm, denoted in the following by $B_H(t)$, is the only Gaussian zero mean, self-similar stochastic process whose covariance function is given by

$$E\left(B_H(t)B_H(s)\right)=\frac{K^2}{2}\left(|t|^{2H}+|s|^{2H}-|t-s|^{2H}\right)$$

where $K^2 = Var(B_H(1))$ and $t,s\in\mathbb{R}^+$. The moving average representation of the fBm reads as

$$B_H(t) = KV_H \int_{\mathbb{R}} \left(|t-x|^{H-\frac{1}{2}} \mathbf{1}_{]-\infty,t]}(s) - |x|^{H-\frac{1}{2}} \mathbf{1}_{]-\infty,0]}(s) \right) dW(x) \quad (1)$$

where $V_H = \left(\Gamma(2H+1)\sin(\pi H)\right)^{1/2} / \Gamma\left(H+\frac{1}{2}\right)$ is a normalizing factor, $\mathbf{1}$ is the indicator function and $W(\cdot)$ denotes the ordinary Brownian motion. The variance of the increment process reads as

$$E\left(\left\{B_H(t) - B_H(s)\right\}^2\right) = K^2 |t-s|^{2H}. \quad (2)$$

The pointwise Hölder exponent[1] of the fBm equals H almost surely at any point t. This value tunes the degree of dependence of the process: $H = 0.5$ reduces to the Brownian motion, whereas values larger (lower) than 0.5 indicate persistence (antipersistence), the stronger the more H deviates from 0.5.

Since a constant pointwise regularity can be too restrictive in many contexts, the fBm can be generalised into the *multifractional Brownian motion* (mBm) ([3], [4]). This is defined replacing the parameter H by a Hölderian function $H(t)$. Relation (1) becomes

$$X_{H(t)}(t) = KV_{H(t)} \int_{\mathbb{R}} \left(|t-x|^{H(t)-\frac{1}{2}} \mathbf{1}_{]-\infty,t]}(s) - |x|^{H(t)-\frac{1}{2}} \mathbf{1}_{]-\infty,0]}(s) \right) dW(x) \quad (3)$$

and the pointwise Hölder exponent equals almost surely $H(t)$ at any point t. This means that the process is asymptotically self-similar with parameter $H(t)$ in a neighborhood of t, in the sense stated by [4]. Denoted by $Y(t,au)=X_{H(t+au)}(t+au) - X_{H(t)}(t)$ the increment process and by $\stackrel{d}{=}$ the equality in distribution, it holds

$$\lim_{a \to 0^+} a^{-H(t)} Y(t+au) \stackrel{d}{=} B_{H(t)}(u), \quad u \in \mathbb{R} \quad (4)$$

The last fundamental equality states that at any point t there exists an fBm with parameter $H(t)$ tangent to the mBm. This allows to deduce the counterpart of relation (2) for the mBm

$$E\left(\left\{X_{H(t)}(t) - X_{H(s)}(s)\right\}^2\right) = K^2 |t-s|^{2H(t)} \quad (5)$$

for sufficiently small $|t-s|$.

Exploiting the suggestion by Papanicolaou and Sølna [6] – who replace the deterministic function $H(t)$ of the mBm by a stationary stochastic process with smooth paths and decaying correlation function independent on W – Ayache and Taqqu [2] introduce the *Multifractional Processes with Random Exponent* (MPRE). Their construction considers (a) the Gaussian field $\{B_H(t)\}_{(t,H) \in [0,1]\times[a,b] \subset (0,1)}$ depending on t and H, with integral representation provided by (1) and (b) the stochastic process $\{S(t)\}_{t \in [0,1]}$ with values in the arbitrary fixed compact interval $[a,b]$. Equipped with this notation, the MPRE of parameter $\{S(t)\}_{t \in [0,1]}$ is defined as

$$Z(t,\omega) = f_2\left(f_1(t)\right) = B_{S(t,\omega)}(t,\omega). \quad (6)$$

Any trajectory of $Z(t,\omega)$ is the composition of two functions:

- f_1: $[0,1] \to [0,1]\times[a,b]$, $t \to (t, S(t,\omega))$, that builds the random process which serves as functional parameter, and
- f_2: $[0,1]\times[a,b] \to \mathbb{R}$, $(t,H) \to B_H(t,\omega)$, that rules the resulting process.

The functional parameter $\{S(t,\omega)\}$ is not necessarily stationary nor independent from the white noise W in (3); when independence is assumed, a particularly relevant result is that the increments of the MPRE form a stationary sequence. Also the continuity of the MPRE ultimately relies on the continuity of the process that provides its random exponent.

We now prove in a very simple way that the marginal distribution of the mBm-MPRE displays high peaks and fat tails. To this aim it suffices to consider the case of just two different Hölder exponents, that is the queueing of two fractional Brownian motions with different parameters.

More precisely, sampling the process at n points on the unit interval so that $t_i = i/n$ ($i = 0,\ldots, n$) and recalling relation (2), set $H(t)$ as piecewise constant

$$H(t_i) = H_0 \mathbf{1}^{(i)}_{[0,p]} + H_1 \mathbf{1}^{(i)}_{(p,1]}, \quad p \in (0,1).$$

Setting $\sigma_0^2 = K^2 (n-1)^{-2H_0}$ and $\sigma_1^2 = K^2 (n-1)^{-2H_1}$, the variance of the marginal distribution trivially reads as

$$\sigma^2 = p\sigma_0^2 + (1-p)\sigma_1^2$$

(remind that here we are not considering the combination of the two sequences but just the queueing), while the density can be written as

$$f_X(x) = \frac{p}{\sigma_0\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_0^2}} + \frac{1-p}{\sigma_1\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_1^2}}. \quad (6)$$

Step 1 (*leptokurtosis*)

Let us prove that $f_X(x)$ is leptokurtic (with respect to the Gaussian distribution). To this aim consider the index of kurtosis excess $\gamma_2 = \mathrm{E}X^4/\sigma^4 - 3$. Values of γ_2 larger than zero denote that the kurtosis is larger that the one of the Gaussian distribution. Since σ is known once p has been fixed, let us calculate $\mathrm{E}X^4$ by the moment-generating function of (6):

$$m_X(t) = E(e^{tX}) = \int_{\mathbb{R}} e^{tx} f_X(x)dx = \int_{\mathbb{R}} \left(\frac{p}{\sigma_0\sqrt{2\pi}} e^{-\frac{x^2 - 2\sigma_0^2 tx}{2\sigma_0^2}} + \frac{1-p}{\sigma_1\sqrt{2\pi}} e^{-\frac{x^2 - 2\sigma_1^2 tx}{2\sigma_1^2}} \right) dx$$

Notice that, for $j = 0, 1$

$$x^2 - 2\sigma_j^2 tx = x^2 - 2\sigma_j^2 tx + \sigma_j^4 t^2 - \sigma_j^4 t^2 = (x - \sigma_j^2 t)^2 - \sigma_j^4 t^2.$$

[1]The Hölder exponent of the function f measures the degree of irregularity of its graph. If there exist a constant C and a polynomial P_n of degree $n<h$ such that $|f(x) - P_n(x-x_0)| \le C|x-x_0|^h$, the Hölder exponent $H(x_0)$ is defined as the supremum of all h's such that the above relation holds.

By substituting in the moment-generating function one gets

$$\int_{\mathbb{R}} \left(\frac{p}{\sigma_0\sqrt{2\pi}} e^{-\frac{\left(x^2-\sigma_0^2 t\right)^2}{2\sigma_0^2}+\frac{\sigma_0^2 t^2}{2}} + \frac{1-p}{\sigma_1\sqrt{2\pi}} e^{-\frac{\left(x^2-\sigma_1^2 t\right)^2}{2\sigma_1^2}+\frac{\sigma_1^2 t^2}{2}} \right) dx =$$

$$= pe^{\frac{\sigma_0^2 t^2}{2}} \int_{\mathbb{R}} \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{\left(x^2-\sigma_0^2 t\right)^2}{2\sigma_0^2}} dx + (1-p)e^{\frac{\sigma_1^2 t^2}{2}} \int_{\mathbb{R}} \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{\left(x^2-\sigma_1^2 t\right)^2}{2\sigma_1^2}} dx$$

The two integrands in the last line are normal densities respectively with mean $\mu_0 = \sigma_0^2 t$ and $\mu_1 = \sigma_1^2 t$. Therefore, the integrals equal one. It follows that

$$m_X(t) = pe^{\frac{\sigma_0^2 t^2}{2}} + (1-p)e^{\frac{\sigma_1^2 t^2}{2}}$$

from which, recalling that $\frac{d^k m}{dt^k}(0) = \mathrm{E}(X^k)$, it follows

$$\frac{d^4 m}{dt^4}(0) = \mathrm{E}(X^4) = 3p\sigma_0^4 + 3(1-p)\sigma_1^4$$

Substituting in the index γ2 and solving the inequality

$$\frac{3p\sigma_0^4 + 3(1-p)\sigma_1^4}{\left(p\sigma_0^2 + (1-p)\sigma_1^2\right)^2} - 3 > 0$$

by trivial algebra it is straightforward to get

$$\left(\sigma_0^2 - \sigma_1^2\right)^2 (p - p^2) > 0 \cdot$$

Obviously, the last inequality is true for $p \in (0,1)$. Since the index of kurtosis is larger than zero, the marginal distribution is leptokurtic with respect to the normal law.

Step 2 (*fat tails*)

In order to prove that the marginal distribution displays also fat tails with respect to the Gaussian density, it suffices to define the ratio

$$Q(x) = \frac{\frac{p}{\sigma_0\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_0^2}} + \frac{1-p}{\sigma_1\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_1^2}}}{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}} =$$

$$= \frac{\sigma}{\sigma_0\sigma_1} \left(p\sigma_1 e^{\frac{\left(\sigma_0^2-\sigma^2\right)x^2}{2\sigma^2\sigma_0^2}} + (1-p)\sigma_0 e^{\frac{\left(\sigma_1^2-\sigma^2\right)x^2}{2\sigma^2\sigma_1^2}} \right)$$

Since it is $\min(\sigma_0^2, \sigma_1^2) \le \sigma^2 \le \max(\sigma_0^2, \sigma_1^2)$, one of the two alternative cases holds

$$Q(x)\Big|_{\sigma_0=\max(\sigma_0,\sigma_1)} = \frac{\sigma}{\sigma_0\sigma_1} \left(p\sigma_1 e^{\frac{\left(\sigma_0^2-\sigma^2\right)x^2}{2\sigma^2\sigma_0^2}} + (1-p)\sigma_0 e^{\frac{-\left|\sigma_1^2-\sigma^2\right|x^2}{2\sigma^2\sigma_1^2}} \right)$$

$$Q(x)\Big|_{\sigma_1=\max(\sigma_0,\sigma_1)} = \frac{\sigma}{\sigma_0\sigma_1} \left(p\sigma_1 e^{\frac{-\left|\sigma_0^2-\sigma^2\right|x^2}{2\sigma^2\sigma_0^2}} + (1-p)\sigma_0 e^{\frac{\left(\sigma_1^2-\sigma^2\right)x^2}{2\sigma^2\sigma_1^2}} \right)$$

and hence

$$\lim_{x\to\pm\infty} Q(x)\Big|_{\sigma_0=\max(\sigma_0,\sigma_1)} = \lim_{x\to\pm\infty} Q(x)\Big|_{\sigma_1=\max(\sigma_0,\sigma_1)} = +\infty \cdot \qquad (7)$$

Relation (7) states that the function at the numerator of $Q(x)$ (i.e. the density obtained by queueing the sequences) tends to zero more slowly than the function at the denominator (i.e. the normal distribution) as x diverges, and this trivially indicates fat tails.

Figure 1 displays a sample path of MPRE: the random functional parameter is shown in panel (a); panel (b) displays the trajectory of the MPRE; panel (c) reproduces the process increments. Notice the (clustered) bursts of variance which produce the fat tails in the distribution of the increments (Figure 2). It is worthwhile to underline the behaviour of the sample autocorrelation function of the surrogated series (Figure 3): in the same way as the actual financial time series, the MPRE is characterized by the absence of linear dependence in its increments but by significant dependence in its squared (absolute) increments.

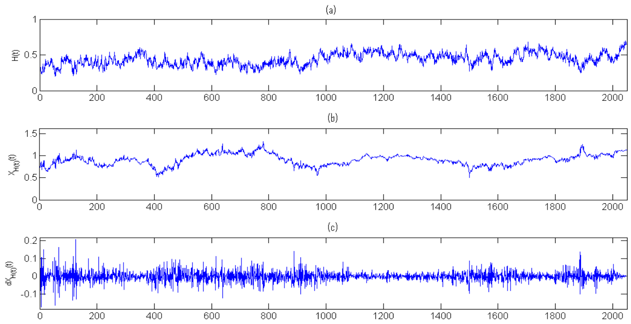

Figure 1. Example of MPRE

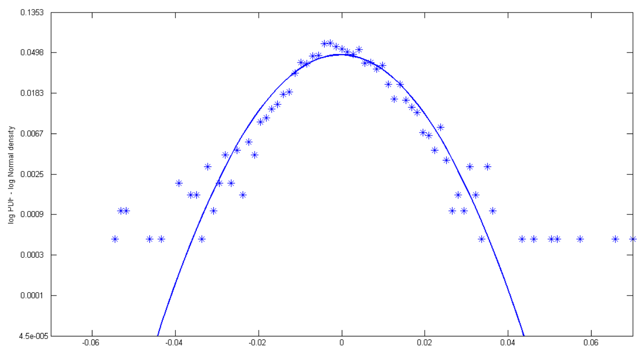

Figure 2. Log-pdf of the MPRE increments compared with

the normal distribution

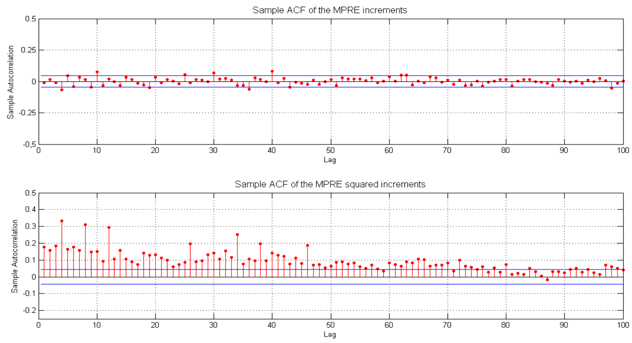

Figure 3. Sample autocorrelation function of the MPRE

III. Estimation of $S(t,\omega)$ and Analysis of Residuals

It is quite evident that the knowledge of the random functional parameter is essential in order to conveniently employ the MPRE in financial modeling. Per se, the functional parameter $S(t,\omega)$ provides a measure of the process Hölder pointwise regularity and therefore it can be viewed as a proxy of the weight that investors assign to the past prices in taking their trading decisions. According to this interpretation, $S(t,\omega)=0.5$ means absence of memory, i.e. lack of influence of the past prices; differently, a positive dependence (no matter the direction of the trend) occur when $S(t,\omega)>0.5$ and it is the stronger the higher $S(t,\omega)$; finally, a negative (or antipersistent) dependence, the stronger the lower $S(t,\omega)$, takes place when $S(t,\omega)<0.5$. Recent analyses ([7], [8], [12] and [13]) show that the functional parameter fluctuates around 0.5, rarely overstays for long periods above this threshold and displays on the contrary large downward movements when market crashes occur. This behaviour seems to be strongly consistent with the logical operation of actual financial markets. In fact, if confidence in the past is a gradually increasing process, the destroy of memory is likely to be a sudden event, often caused by shocking innovations (such as, for example, a destabilizing political news or the release of negative financial reports).

With these premises in mind, the estimation of $S(t,\omega)$ plays a fundamental role in the tangible use of the MPRE. Roughly speaking, it is about estimating the trajectory of panel (a) of Figure 1 by knowing the sample path of the process (panel (b)). In order to accomplish this step we can use the estimator introduced in [7] and recently improved in [13], contributions to which we refer for a detailed description of the estimator. Here, we will just recall that the estimator – written for the q-lagged increments of a series $\{X_{j,n}\}$ with n data sampled in discrete time on the grid $t = 1,\ldots, n$ and with unit time variance equal to K^2 – is referred to a window of length δ and reads as

$$H^k_{\delta,q,n,K}(t) = \frac{\log\left(\frac{\sqrt{\pi}}{\delta-q+1}\sum_{j=t-\delta}^{t-1}\left|X_{j+q,n}-X_{j,n}\right|^k / 2^{k/2}\Gamma\left(\frac{k+1}{2}\right)K^k\right)}{k\log\left(\frac{q}{n-1}\right)} \quad (8)$$

The moving-window estimator (8):

- assumes the increments $X_{j+q,n} - X_{j,n}$ of the series to be normally distributed within the window δ with mean zero and variance given by relation (5);
- is normally distributed with mean equal to the parameter to be estimated and variance that can be explicitly calculated when $H(t) = 0.5$. Confidence intervals for the each δ and $H(t)$ are provided by Monte Carlo simulations in [13];
- has a $(\sqrt{\delta}\log n)$-rate of convergence, δ and n respectively being the length of the estimation window and the number of sampling points.

An idea of how estimator (8) works is provided in Figure 4: the thin blue line indicates the functional parameter to be estimated and the thick red line indicates the estimates

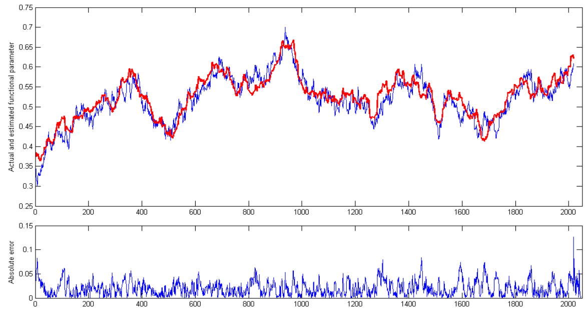

Figure 4. Actual and estimated functional parameter of MPRE

obtained with δ=30. The down panel displays the absolute errors; the overall mean absolute error is 0.0199.

Once the estimation of $S(t,\omega)$ has been obtained by (8), we use it to calculate the residuals in the usual way

$$r_t = \frac{dX_{t,n}}{K(n-1)^{-H^k_{\delta,q,n,K}(t)}}. \quad (9)$$

If the model and the estimator work, we obviously expect r_t to be i.i.d. standard normal. We check this result using the Jarque-Bera normality test [16] and evaluating the sample autocorrelation functions of both the residuals and squared residuals. The Jarque-Bera test quantifies the distance from the Gaussian distribution in terms of skewness and kurtosis and then computes a single p value using the sum of these discrepancies.

It is well-known that heavy tails and volatility clustering, the very motivations for the use for example of GARCH models, are expected to disappear once the returns are normalized by the level of volatility, making their distribution Gaussian. Nevertheless, most studies employing ARCH and GARCH models document the existence of severe excess kurtosis in the estimated residuals [25] and show extremal dependence, whose strength reduces only as extreme levels increase. This unpleasant result, often ascribed to model misspecifications, structural changes or outliers, constitutes a serious warning about the capability of such models to capture entirely the variation in volatility; it is generally addressed by modeling the residuals by some *ad hoc* distribution. The most popular distributions used to this aim are the (eventually asymmetric) Student's t [19], [20], [21], [22], the generalized Pareto [17], [18], the normal inverse Gaussian distribution [24] or the double exponential, as a particular case of the generalized error distribution when the tail thickness parameter equals 1 [23]. In spite of the many efforts that have been devoted to the analysis of this topic, nevertheless the choice of such distributions appears quite arbitrary; in authors' opinion, it makes questionable the capability of the above recalled models to capture the volatility dynamics and motivates the next Section, in which the behaviour of the MPRE residuals is analysed.

IV. EMPIRICAL APPLICATION

WITH REGARD TO THE LAST FIVE YEARS (01/02/2005-06/30/2010), WE analysed the four main financial stock indices, whose distributional parameters are summarized in Table 1. No discussion about the highly non Gaussian behaviour of the the log-increments marginals: here we just underline that the "worst" behaviour in terms of kurtosis is represented by the Dow Jones, mainly because it is the most impacted index by the 2008 crisis.

TABLE1. DATA

	Distribution parameters					
	Obs.	***Mean***	***St.Dev.***	***Kurtosis***	***Skewness***	***Tail-index***
DJIA	1,379	0.0000	0.0137	12.8018	0.0246	0.2252
FTSE	1,384	0.0000	0.0139	11.1170	-0.1084	0.2417
N225	1,342	-0.0001	0.0172	11.3617	-0.4255	0.2161
HSI	1,374	0.0003	0.0187	11.2096	0.0838	0.2507

Using relation (8), we have estimated the pointwise regularity of each index, setting $\delta = 15$ and q=1 (the right K was estimated using the technique proposed in [13]). The estimates of $H^k_{\delta,q,n,K}(t)$ are displayed in Figure 5. Notice that in all the cases the pointwise regularity fluctuates around 0.5, the sole value consistent with the principle of arbitrage. The sudden drops in the parameter, observed in correspondence of financial crashes, are followed by more gradual upward movements, consistently with the financial interpretation shortly discussed in the previous Section.

The second step of the empirical analysis consisted in calculating the residuals (9). Table II summarizes their distributional properties. Due to the nonlinear transformation induced by the estimator (8) on the log-variations, the mean differs from the original value but is still close to zero. While the skewness is still far from zero (even if, as it will be seen in a while, the difference is not significant), notice that the standard deviation is approximately one and the kurtosis is very close to 3, values of the $N(0,1)$ law.

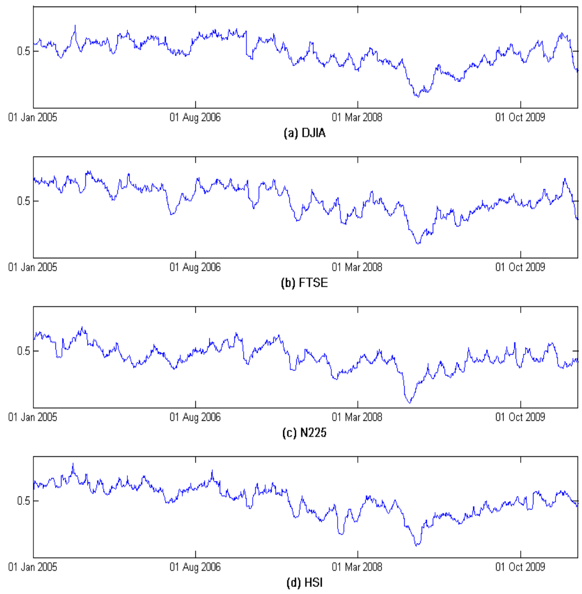

Figure 5. Pointwise regularity estimates of the four indices

TABLE2. RESIDUALS

	Distribution parameters					
	Obs.	***Mean***	***St.Dev.***	***Kurtosis***	***Skewness***	***Tail-index***
DJI	1,379	0.0314	0.9952	2.9393	-0.2571	-0.3349
FTSE	1,384	0.0286	1.0120	2.7483	-0.0233	-0.2948
N225	1,342	0.0073	1.0092	2.7750	-0.2172	-0.3408
HSI	1,374	0.0450	0.9984	2.8890	-0.1547	-0.2913

In order to check for the normality of the residuals we performed the Jarque-Bera test for increasing time spans (from 20 trading days to about four trading years, i.e. 1,000 observations). The results, summarized in Tables III, show that – except for the Dow Jones, which is normally distributed at α=0.05 for about one trading year – all the other indices have residuals which are standard normal for the overall period. The results are compared with those obtained fitting the time series by GARCH(1,1) (Table IV). Notice that for almost all the sizes the MPRE residuals perform better than the corresponding GARCH residuals.

The last analysis concerned the autocorrelation of both the residuals and the squared residuals. Figures 6 and 7 display the results: as expected, in all the cases the residuals

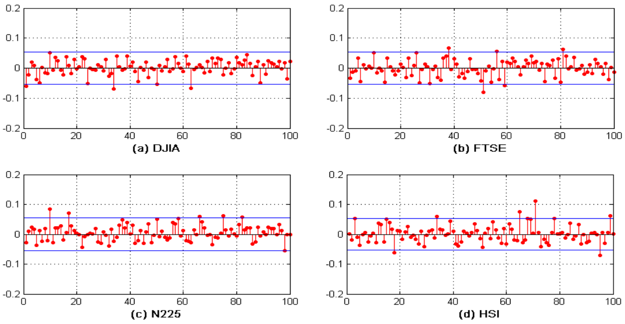

Figure 6. Sample autocorrelation functions of the residuals

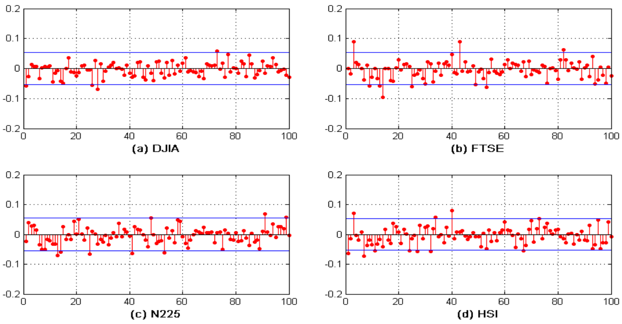

Figure 7. Sample autocorrelation functions of the squared Residuals

TABLE3. JARQUE-BERA TEST FOR MPRE RESIDUALS

	p-values			
Size	**DJIA**	**FTSE**	**N225**	**HSI**
20	0.5000	0.5000	0.5000	0.4160
40	0.4813	0.5000	0.2610	0.5000
60	0.5000	0.2053	0.5000	0.1108
80	0.5000	0.5000	0.5000	0.5000
100	0.5000	0.5000	0.5000	0.5000
120	0.1696	0.5000	0.5000	0.5000
140	0.1321	0.3452	0.4224	0.4818
160	0.1055	0.2685	0.3053	0.5000
180	0.0765	0.1650	0.5000	0.5000
200	0.0681	0.1312	0.5000	0.3973
220	0.0513	0.1479	0.4290	0.2428
240	0.0462	0.1506	0.4505	0.1929
260	0.0371	0.1137	0.2606	0.1513
280	0.0275	0.0869	0.2469	0.1462
300	0.0237	0.0727	0.1626	0.1750
320	0.0286	0.0665	0.2115	0.3160
340	0.0217	0.0516	0.2099	0.2920
360	0.0239	0.0809	0.1561	0.2781
380	0.0430	0.0747	0.2189	0.1908
400	0.0344	0.0936	0.1496	0.1646
420	0.0434	0.0812	0.1159	0.1735
440	0.0382	0.1325	0.1124	0.1726
460	0.0496	0.1304	0.1301	0.1689
480	0.0676	0.1068	0.1331	0.2244
500	0.0380	0.2167	0.1420	0.1591
520	0.0443	0.2228	0.1804	0.1296
540	0.0424	0.1726	0.1838	0.0986
560	0.0390	0.1432	0.1403	0.1126
580	0.0316	0.1295	0.1328	0.1061
600	0.0310	0.1443	0.1059	0.1077
620	0.0539	0.1547	0.0955	0.0927
640	0.0437	0.1782	0.0575	0.1169
660	0.0334	0.1507	0.0489	0.1224
680	0.0317	0.1312	0.0442	0.1069
700	0.0309	0.1046	0.0367	0.0863
720	0.0200	0.0886	0.0434	0.0662
740	0.0203	0.0867	0.0347	0.1128
760	0.0181	0.0784	0.0300	0.1156
780	0.0176	0.0617	0.0268	0.0917
800	0.0137	0.0610	0.0257	0.1021
820	0.0106	0.0503	0.0238	0.1179
840	0.0088	0.0427	0.0207	0.1462
860	0.0071	0.0370	0.0146	0.1801
880	0.0043	0.0224	0.0114	0.1388
900	0.0039	0.0235	0.0072	0.1234
920	0.0040	0.0421	0.0071	0.1041
940	0.0031	0.0352	0.0060	0.0918
960	0.0038	0.0471	0.0077	0.1145
980	0.0036	0.0302	0.0074	0.1120
1,000	0.0034	0.0219	0.0068	0.0858

TABLE4. JARQUE-BERA TEST FOR GARCH(1,1) RESIDUALS

	p-values			
Size	**DJIA**	**FTSE**	**N225**	**HSI**
20	0.5000	0.5000	0.5000	0.4395
40	0.1907	0.5000	0.1888	0.1741
60	0.2972	0.1220	0.5000	0.0246
80	0.1105	0.2492	0.5000	0.5000
100	0.0367	0.4823	0.5000	0.5000
120	0.0070	0.2284	0.5000	0.5000
140	0.0045	0.1182	0.5000	0.5000
160	0.0059	0.0953	0.5000	0.4262
180	0.0038	0.0328	0.5000	0.0210
200	0.0042	0.0324	0.5000	0.0277
220	0.0037	0.0592	0.5000	0.0239
240	0.0027	0.0525	0.5000	0.0276
260	0.0020	0.0509	0.1887	0.0355
280	0.0016	0.0410	0.2006	0.0436
300	0.0014	0.0326	0.1055	0.0590
320	0.0019	0.0291	0.1547	0.1068
340	0.0014	0.0211	0.1334	0.1330
360	0.0015	0.0233	0.1319	0.1920
380	0.0031	0.0148	0.2071	0.2755
400	0.0030	0.0180	0.1396	0.3451
420	0.0026	0.0059	0.0897	0.4164

	p-values			
Size	**DJIA**	**FTSE**	**N225**	**HSI**
440	0.0029	0.0091	0.0947	0.4419
460	0.0041	0.0078	0.0293	0.4050
480	0.0041	0.0057	0.0335	0.3607
500	0.0025	0.0104	0.0456	0.5000
520	0.0041	0.0085	0.0567	0.5000
540	0.0031	0.0134	0.0719	0.5000
560	0.0018	0.0193	0.0739	0.3073
580	0.0018	0.0048	0.0740	0.2920
600	0.0014	0.0031	0.0634	0.1933
620	0.0026	0.0033	0.0618	0.1960
640	0.0020	0.0039	0.0289	0.1314
660	0.0031	0.0010	0.0248	0.0509
680	0.0027	0.0010	0.0312	0.0512
700	0.0032	0.0010	0.0289	0.0694
720	0.0010	0.0010	0.0356	0.0662
740	0.0010	0.0010	0.0068	0.0687
760	0.0010	0.0010	0.0068	0.0718
780	0.0010	0.0010	0.0050	0.0317
800	0.0010	0.0010	0.0042	0.0329
820	0.0010	0.0010	0.0041	0.0266
840	0.0010	0.0010	0.0030	0.0264
860	0.0010	0.0010	0.0010	0.0155
880	0.0010	0.0010	0.0010	0.0092
900	0.0010	0.0010	0.0010	0.0080
920	0.0010	0.0010	0.0010	0.0107
940	0.0010	0.0010	0.0010	0.0011
960	0.0010	0.0010	0.0010	0.0010
980	0.0010	0.0010	0.0010	0.0010
1,000	0.0010	0.0010	0.0010	0.0010

are not autocorrelated.

V. CONCLUSION AND FURTHER DEVELOPMENTS

In this paper we have proposed to model the financial time series using a very versatile stochastic process, the MPRE, recently introduced by [2]. In author's knowledge it is the very first time that this process is used in finance and our preliminary findings, in terms of well-behaved distributions (of the increments and of the squared ones) as well as the goodness of fit of the sample autocorrelation functions, indicate that the model really deserves more attention. After discussing the main properties of the model and proving that the variability of its pointwise regularity exponent produces high-peaked and fat tailed marginal distributions, we use a robust absolute moment-based estimator of the MPRE functional parameter in order to quantify the Hölderian pointwise regularity of four main stock indices. We find that the estimated parameters consistently describe the market behaviour (we sketch a financial intuition of this in Section III) and explain the whole volatility process. This can be seen by analyzing the residuals of the historical data filtered by $H^k_{\delta,q,n,K}(t)$. Since we find evidence that such residuals are i.i.d. standard normal until a time horizon of about one trading year in the worst case of DJIA, of four trading years for the FTSE and for the whole sample for the N225 and the HSI, we conclude that the model succeeds in capturing locally the volatility process better than ARCH/GARCH-like models. These generally need to resort to *ad hoc* fat tailed distributions to fit their residuals. Due to limits of space, the last conclusion is just sketched in this paper and we plan to deepen the topic in a future dedicated work.

ACKNOWLEDGMENT

The authors wish to thank three anonymous referees for useful comments on an earlier version of this paper.

REFERENCES

[1] A. Greenspan, "Discussion at Symposium: Maintaining Financial Stability in a Global Economy," the Federal Reserve Bank of Kansas City, (1997), 54.

[2] A. Ayache, M. Taqqu, "Multifractional processes with random exponent", Publicacions Matemàtiques, 49, 2005, pp.459–486.

[3] R. Péltier, J. Lévy Véhel, "Multifractional brownian motion: definition and preliminary results", Technical Report RR-2645, INRIA, 1995.

[4] A. Benassi, S. Jaffard, and D. Roux, "Elliptic gaussian random processes", Rev. Mat. Iberoamericana, 13, 1997, pp.19–90.

[5] B. Mandelbrot and J. Van Ness, "Fractional brownian motions, fractional noises and applications", SIAM Review, 10(4), 1968, pp.422–437.

[6] G.C. Papanicolaou and K. Sølna, "Wavelet based estimation of local Kolmogorov turbulence", in *Theory and applications of long-range dependence*, Birkhäuser Boston, Boston, MA, 2003, pp.473–505.

[7] S. Bianchi, "Pathwise identification of the memory function of multifractional brownian motion with application to finance", International Journal of Theoretical and Applied Finance (IJTAF), 8(2), 2005, pp.255–281.

[8] S. Bianchi and A. Pianese, "Multifractional properties of stock indices decomposed by filtering their pointwise Hölder regularity", International Journal of Theoretical and Applied Finance (IJTAF), 11(6), 2008, pp.567–595.

[9] R.L. Costa and G. L. Vasconcelos, "Long-range correlations and nonstationarity in the brazilian stock market", Physica A: Statistical Mechanics and its Applications, 329(1-2), 2003, pp.231–248.

[10] A.F. Crepaldi, C.R. Neto, F.F. Ferreira and G. Francisco, "Multifractal regime transition in a modified minority game model", Chaos, Solitons & Fractals, 42(3), 2009, pp.1364–1371.

[11] A. Pantanella and A. Pianese, "Minimum risk portfolios using MMAR", Proceedings of the 10th WSEAS international conference on Mathematics and computers in business and economics, World Scientific and Engineering Academy and Society (WSEAS), 2009, pp.64-71.

[12] S. Bianchi, A. Pantanella, "Pointwise Regularity Exponents and Market Cross-Correlation", International Review of Business Research Papers, 1(1), 2010, pp.39-51.

[13] S. Bianchi, A. Pantanella and A. Pianese, "Modeling stock prices by multifractional Brownian motion: an improved estimation of the pointwise regularity", *submitted*

[14] R. Cont, "Empirical properties of asset returns: stylized facts and statistical issues". Quantitative Finance, 1(2), 2001, pp.223–236.

[15] M. Sewell, "Characterization of Financial Time Series", preprint, 2008

[16] C.M. Jarque and A.K. Bera, "A test for normality of observations and regression residuals", International Statistical Review 55, 1987, pp.163–172.

[17] F. Laurini and J.A. Tawn, "Regular Variation and Extremal Dependence of GARCH Residuals with Application to Market Risk Measures", Econometric Reviews, 28(1-3), 2009 , pp.146-169.

[18] McNeil, A.J., and R. Frey, (2000), Estimation of tail-related risk measures for heteroscedastic financial time series: an extreme value approach. Journal of Empirical Finance, 7, pp.271-300.

[19] T. Bollerslev, A Conditionally Heteroskedastic Time Series Model For Speculative Prices and Rates of Return, *Review of Economics and Statistics* 69, 1987, pp.542–547.

[20] T. Bollerslev, R. Chou and K. Kroner, ARCH modelling in finance: a review of theory and empirical evidence, Journal of Econometrics, 52, 1992, pp.5-60.

[21] P. Hall and Q. Yao, Inference in ARCH and GARCH models with heavy-tailed errors, Econometrica, 71, 2003, pp.285-317

[22] C. Fernandez and M. Steel, On Bayesian Modeling of Fat Tails and Skewness, Journal of the American Statistical Association, 93, 1998, pp.359-371.

[23] D.B. Nelson, "Conditional Heteroskedasticity in Asset Returns: a New Approach", Econometrica, 59(2), 1991, pp.347-370.

[24] J.H. Venter and P.J. de Jongh, Risk Estimation Using the Normal Inverse Gaussian Distribution, Journal of Risk, 4, (2002), pp.1-23.

[25] R. Bali and H. Guirguis, Extreme observations and non-normality in ARCH and GARCH, International Review of Economics & Finance, 16, 2007, pp.332-346