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Bayesian Inference of the GARCH model with Rational Errors

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Abstract. We propose to use Padé approximants which construct rational functions for the error terms of the GARCH model. First in order to perform the Bayesian inference we develop a Markov chain Monte Carlo method for the GARCH model with the rational errors (GARCH-RE). The Markov chain Monte Carlo method is performed by the Metropolis-Hastings algorithm with the Student's t-distribution. We confirm that the Metropolis-Hastings algorithm with the Student's t-distribution is an efficient method for the Bayesian inference of the GARCH-RE model. We then empirically analyze the GARCH-RE model with USD/JPY exchange rate return data and find that the GARCH-RE model is superior to the GARCH model with normal errors. Thus the GARCH-RE model is considered to be an alternative GARCH-type model for financial time series analysis.

Keywords: Time series analysis, Markov Chain Monte Carlo, Bayesian inference, GARCH model, Pade approximants

1. Introduction

In financial applications volatility plays a central role for risk measurement of financial assets. It has been recognized that volatility of asset returns changes through time. Such time-varying volatility is well modelled by the GARCH model[1]. In the GARCH model asset returns r_t are expressed as $r_t = \sigma_t \varepsilon_t$, where σ_t is the time-changing volatility which is a function of past returns and volatilities. ε_t is an independent and identically distributed random variable with mean 0 and variance 1. In the original GARCH model the normal distribution was used for ε_t errors.

It is well known that the unconditional return distributions show fatter tails than the normal distributions. Although the GARCH model with the normal errors (GARCH-N) also shows the fat tail distributions, it is recognized that the model does not sufficiently account for the leptokurtosis of the financial return data. One way to circumvent the problem is the introduction of fat tail distributions for ε_t errors such as student t-distribution[2]. Using fat-tailed distributions for ε_t errors the non-normality of the model can be improved. However there is no consensus for the distributional assumption of ε_t errors. One could also choose other fat-tailed distributions.

In this paper we apply Padé approximants which consist of rational functions for ε_t errors. The Pade approximants are flexible to approximate a function in a certain domain. In Ref.[3] Padé approximants are used to describe the interest rate return distributions. The empirical interest rate return distributions at tails are usually fatter than the normal distribution. This fat-tailed nature is successfully described by a rational function from the Padé approximants. Here we apply the Padé approximants for the error distribution of the GARCH model and empirically investigate whether such a model performs better than the standard GARCH model with normal errors.

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In order to infer model parameters from financial data we employ the Markov Chain Monte Carlo (MCMC) method based on the Bayesian inference. There have been various algorithms proposed to implement the MCMC method for GARCH models. In this study we use the Metropolis-Hastings algorithm[4,5] with an adaptive Student's t-proposal density which has been shown to be effective for GARCH-type models[6, 7, 8, 9, 10,11,12]. We develop this algorithm for the GARACH model with the rational errors (GARCH-RE) and then make an empirical study with USD/JPY exchange rate returns by the GARCH-RE model.

2. GARCH Model

The GARCH(l,m) model[1] is defined by

$$y_t = \sigma_t \varepsilon_t \,, \tag{1}$$

where y_t corresponds to an asset return and

$$\sigma_t{}^2 = \omega + \sum_{i=1}^{l} \alpha_i y_{t-i}^2 + \sum \beta_i \sigma_{t-i}^2 \,, \tag{2}$$

where the GARCH parameters are restricted to $\omega > 0$, $\alpha_i \geq 0$ and $\beta_i \geq 0$ to ensure a positive volatility, and the stationary condition $\sum_{i=1}^{l} \alpha_i + \sum_{i=1}^{m} \beta_i < 1$ is also required. The error term ε_t is an independent variable generated by a specific probability distribution. Our main idea is to use a rational error distribution for the error term ε_t. In this study we focus on GARCH(1,1) model where the volatility process σ_t^2 is given by

$$\sigma_t{}^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 \,, \tag{3}$$

and we simply denote it the GARCH model.

3. Rational Error Distribution

We use Padé approximants for the probability distribution of the error term ε_t. Padé approximants $P_{M,N}(x)$ are given as a rational function of variable *x* with two polynomials $T_M(x)$ and $B_N(x)$,

$$P_{M,N}(x) = \frac{T_M(x)}{B_N(x)}. \tag{4}$$

M and N stand for the degrees of the polynomial $T_M(x)$ and $B_N(x)$ respectively. Since $P_{M,N}(x)$ is supposed to be a probability distribution, it must be positive. We further impose on the conditions that $P_{M,N}(x)$ takes a maximum value at the origin and is symmetric to the *x*=0 axis, i.e. $P_{M,N}(x) = P_{M,N}(-x)$. In Ref.[3] possible normalizable distributions with finite variances are derived to approximate the interest rate distributions. The simplest normalized distribution with tunable parameters is given by

$$P_{0,4}(x) = \frac{a_1}{\pi(1 + (a_1^2 + 2a_2)x^2 + a_2^2 x^4)} \,, \tag{5}$$

where a_1 and a_2 are tunable parameters. The variance of this $P_{0,4}(x)$ is given by -1/a_2. We set the variance of the error distribution to one. Thus for $P_{0,4}(x)$ we also set $a_2 = -1$. The final form of P(x) with $q \equiv a_1$ is written as

$$P\ (x) = \frac{q}{\pi(1 + (q^2 - 2)x^2 + x^4)} \,. \tag{6}$$

4. Bayesian Inference

The GARCH-RE model has 4 parameters α, β, ω and q which have to be determined so that the model matches the given data, i.e. here financial data. We determine the parameters by the Bayesian inference performed by the MCMC. Let us consider the posterior density of the GARCH-RE model with *n* financial data denoted by y = (y_1, y_2, . . . , y_n). From the Bayes theorem the posterior density π(θ|y) is given by π(θ|y) ∝ L(y|θ) π(θ), where θ≡ (θ_1, θ_2, θ_3, θ_4) = (α, β, ω, *q*) and L(y|θ) stands for the likelihood function of the GARCH-RE model. π(θ) is the prior density for θ. In this study we assume that the prior density π(θ) is constant. The likelihood function of the GARCH model is given by

$$L(y\mid\theta)=\prod_{t=1}^{n}\frac{q}{\pi\sigma_t(1+(q^2-2)y_t^2/\sigma_t^2+y_t^4/\sigma_t^4)}\ . \tag{7}$$

Using the posterior density π(θ|y) the GARCH parameters α, β, ω and the rational error parameter q are inferred as the expectation values as

$$\langle\theta_i\rangle=\frac{1}{Z}\int\theta_i\pi(\theta\mid y)d\theta\,, \tag{8}$$

where Z is a normalization constant, $Z=\int\pi(\theta\mid y)d\theta$.

In order to evaluate eq.(8) we employ the MH algorithm[4,5]. In Refs.[6, 7], a multivariate Student’s t-distribution is used as the proposal density of the MH algorithm for the GARCH parameter estimation and it is shown that the MH algorithm with a multivariate Student’s t-distribution (MH-STD) gives a good performance for GARCH-type models[6,7,8,9,10,11,12]. In this study we also use multivariate Student’s t-distributions for the proposal density of the MH algorithm and examine the effectiveness of the MH-STD algorithm for the GARCH-RE model.

5. Multivariate Proposal Density

Our MH algorithm uses (p-dimensional) multivariate Student’s t-distributions given by

$$g(\theta)=\frac{\Gamma((\nu+p)/2)/\Gamma(\nu/2)}{\det\Sigma^{1/2}(\nu\pi)^{p/2}}\left[1+\frac{(\theta-M)^t\Sigma^{-1}(\theta-M)}{\nu}\right]^{-(\nu+p)/2}\ , \tag{9}$$

where θ and M are column vectors, θ = (θ_1, . . . , θ_p) and M = (M_1, . . . ,M_p), and M_i = E(θ_i). Σ is the covariance matrix defined as

$$\frac{\nu\Sigma}{\nu-2}=E[(\theta-M)(\theta-M)^t]. \tag{10}$$

ν is a parameter to tune the shape of the Student’s t-distribution. Note that when $\nu\to\infty$ the Student’s t-distribution goes to a Gaussian distribution. Since the GARCH-RE model has 4 parameters: θ =(θ_1, θ_2, θ_3, θ_4) = (α, β, ω,q), the dimension p of the Student’s t-distribution is 4.

6. Numerical Test with Artificial Data

In this section we investigate the effectiveness of the MH-STD algorithm for the GARCH model with the rational error by using artificial GARCH data generated with known parameters. First we set the GARCH parameters and the rational error parameter to α = 0.05, β = 0.9, ω = 0.05 and q = 1.8, and generate 5000 data by the GARCH process with these parameters. Then we estimate the values of the parameters only from the data by the Bayesian inference and check whether our MCMC algorithm correctly reproduces the parameter values or not.

We performed the MCMC simulations as follows. In order to estimate the parameters of the Student’s t-distribution first we make a short pilot run by a standard Metropolis algorithm. We discarded the first 5000 samples as burn-in process or thermalization and accumulated 1000 data for estimation of M and Σ. The estimated values of M and Σ are substituted to the Student’s t-distribution g(θ). Then we switch the MCMC algorithm to the MH algorithm with g(θ) and continue the MCMC simulations. Since at the first stage of the simulations the parameters of the Student's t-distribution may not be accurately estimated we re-calculate M and Σ every 1000 updates and substitute the parameters to g(θ). Thus the parameters of g(θ) are updated adaptively during the MCMC simulation. After accumulating 100000 data we evaluate values of the GARCH and rational error parameters. In order to compare the effectiveness of the MH-STD algorithm, we also perform the simulation by a standard Metropolis algorithm.

In Table 1 we list the results of the parameters inferred by the MH-STD algorithm and the Metropolis algorithm. We see that both algorithms well reproduce the values of the input parameters within the standard deviation. This observation simply means that both algorithms worked correctly. However the difference arises in the autocorrelation time. The autocorrelation times of the MH-STD algorithm are much smaller than those of the Metropolis algorithm, which indicates that the performance of the MH-STD algorithm is superior to the Metropolis algorithm.

Table 1: Values of estimated parameters and autocorrelation times. SD and SE stand for standard deviation and statistical error respectively.

	α	β	ω	q
true	0.05	0.9	0.05	1.8
MH+STD	00585	0.885	0.058	1.771
SD	0.0096	0.016	0.010	0.047
2τ	2.0 ± 0.1	2.3 ± 0.1	2.3 ± 0.1	2.0 ± 0.1
Metropolis	0.0585	0.884	0.058	1.769
SD	0.0097	0.016	0.010	0.047
2τ	400 ± 140	770 ± 360	760 ± 340	60 ± 3

7. Empirical Results

In this section we apply the GARCH-RE model for the financial data obtained from the real financial markets and investigate how the model fit to the financial data. We used USD/JPY exchange rate returns from 4 Jan. 1999 to 29 Dec. 2006. As in the previous section we performed the MCMC simulations for the GARCH-RE model by the MH-STD algorithm. For comparison we also performed the simulations for the GARCH model with normal errors. The parameters determined by the MCMC simulations are listed in Table 2. To compare the two models we calculate AIC[13] and DIC[14] criterions which evaluate the goodness-of-fit of the models. Here AIC and DIC are defined so that smaller values indicate better goodness. We find that both AIC and DIC favor the GARCH-RE model (See Table 2).

Table 2: Values of estimated parameters, autocorrelation times, AIC and DIC.

	α	β	ω	q
GARCH-RE	0.043	0.946	0.0127	1.64
SD	0.011	0.015	0.0057	0.06
2τ	6.5 ± 2.9	7.8 ± 1.8	8.6 ± 1.9	10.0 ± 7.5
AIC	1878.26			
DIC	3743.86			
GARCH-N	0.0314	0.940	0.0113	
AD	0.0077	0.017	0.0049	
2τ	4.7 ± 1.3	8.2 ± 2.9	9.5 ± 3.6	
AIC	1904.35			
DIC	3799.52			

8. Conclusion

We proposed to use a rational function for the error term of the GARCH model and constructed the GARCH-RE model. To perform the Bayesian inference of the GARCH-RE model we developed the MH algorithm with the Student's t-distributions and found that by testing the MH algorithm with artificial data the MH-STD algorithm works well for the GARCH-RE model. We applied the GARCH-RE model for USD/JPY exchange rate returns and found that the GARCH-RE model is superior to the GARCH model with normal errors. Therefore the GARCH-RE model may serve as an alternative model of the GARCH-type models. It might be interesting to see whether the GARCH-RE model performs well also for other asset returns.

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