2010 International Conference on Economics, Business and Management
IPEDR vol.2 (2011) © (2011) IAC S IT Press, Manila, Philippines

Pricing Alternatives in Incomplete Markets

An Application for Carbon Allowances

Marius-Cristian Frunza
Sagacarbon, CDC Climat
Structuring
Paris, France
e-mail: marius-cristian.frunza@m4x.org

Dominique Guegan
Sorbonne Paris I University
CES, PSE
Paris, France
e-mail: dguegan@univ-paris1.fr

***Abstract*—In this paper, we bring a new econometric perspective for CO_2 emission prices modelling and we provide with an innovative methodological approach to compute option prices in incomplete markets. We apply our methodology to carbon derivatives. We calibrate several Generalized Hyperbolic and GARCH models with various innovations, using the CO_2 European Union Allowances (EUA) daily prices from 2005 to 2010. The issue of probability changing is handled with a generalization of Duan's (11) theoretical concept of local risk neutrality. Thus we use some modellings features under the historical measure to derive a valuation model for options implemented through an empirical martingale correction approach with variance rescaling. We compare this approach, with one followed by Gerber and Siu (17) using a stochastic discount factor exponential affine and with another one recently developed by Chorro, Guégan and Ielpo (7) considering a pure empirical martingale correction technique. Option prices computed through different techniques for different maturities and moneyness show promising results for the Normal Inverse Gaussian distribution.**

Keywords-component; Carbon, EUA, Generalized Hyperbolic distribution, GARCH modelling, pricing, Incomplete markets, Empirical Martingale Correction

I. INTRODUCTION

According to Kyoto protocol's provisions, the industrialized countries have to reduce the greenhouse gas emissions by 5 percent in the period 2008-2012, with respect to levels in 1990. The protocol imposes the cap and trade system for emission allowances as one of the primary standards that could allow economies to achieve greenhouse gas emission reduction. Thus, the right to pollute is considered to be a tradable asset, and its price is determined by market drivers of supply and demand. This means that a "reasonable" knowledge of the CO_2 price behavior is fundamental for both predictive purpose and pricing accuracy.

Some recent studies Paolella and Taschini (21), Ulrih-Homburg and Wagner (23), Benz and Truk (3), Daskalakis, Psychoyios and Markellos (8), (9) investigated the behavior of European allowances prices on the period 2005 - 2007, underlying the particularities of this market like the non-Gaussian behavior, the auto-regressive phenomena and the presence of the convenience yield. Following this line a primal purpose of our work is to enrich the modeling methods with Generalized Hyperbolic distributions and non-Gaussian GARCH approaches.

Over the last two years the carbon market faced a growing influx of liquidities from financial and industrial players, with investment supports showing a relatively high volatility (45% between 2005 and 2010). Given the need for compliance at a minimum cost the derivative contracts appeared as a necessity and the organized exchanges started to witness a development of derivatives' transactions since early 2007 with more significant volumes for the vanilla options over the last year. Yet compared to other markets like equities or FX, carbon derivatives are still limited in terms of volumes and number of transactions (an average of 5-10 transactions / day cleared on European Climate Exchange). Given the particular case of the carbon allowances market with efficiency and completeness issues (Frunza (15), (16)) and low liquid options market issues the classic methods of risk-neutral pricing under non-arbitrage assumptions one of them being the Black and Scholes theory cannot apply in this environment.

II. UPON RISK NEUTRALITY

In the derivative markets we consider the volatility of the pricing process is different from the historical volatility of the underlying asset process. This occurs because market players will set state prices to reflect their aggregate preferences and views about market expectations at a certain moment. The classic Black-Scholes (2) pricing model distinguishes between historical and market volatility, calibrated on trade options prices. The theoretical passage of the economy from the physical measure $\boldsymbol{P}$ to the business measure $\boldsymbol{Q}$ is insured by Girsanov's theorem, but for being able to apply it options should be liquid and the market should be complete.

In the case of the carbon allowances market it appears impossible to calibrate the pricing process directly on options' prices. Thus we are dealing with a discrete time market that is well known as being in general incomplete. Thus, the business measure $\boldsymbol{Q}$ is not unique and there is a multiplicity of economy rotations that are compatible with the observed prices. For this situation the explicit probabilistic approaches as described by Black-Scholes (2), Gerber-Shiu (17) through the Esscher transform, Eberline-Prause (13) for GH processes or Heston-Nandi (20) for GARCH models are directly doable.

The recent researches mitigate the pricing issues outside the classic Gaussian risk neutral framework by introducing leptokurtic and skewed distributions and extending Bollerslev's GARCH process. The problem of pricing under market completeness challenges was largely described by the literature through different approaches, mainly focusing on the Stochastic Discount Factor (Gourieroux-Monfort (19), Rosenberg-Engle (22), Chorro-Guégan-Ielpo (6)) or developing empirical approaches (Duan-Simonato (12), Duan (11), Barone-Adesi (1), and Eriksson-Forsberg (18)), and Chorro et al (7) for instance. Yet all the quoted studies are tested on underlings like equities or fixed incomes, which by nature are the closest to the classic framework of financial markets.

Changing volatility in real markets makes the perfect replication argument in the sense of Black and Scholes invalid. Markets are then incomplete in the sense that perfect replication of contingent claims using only the underlying asset and a riskless bond is impossible. Of course markets become complete if a sufficient (possibly infinite) number of contingent claims are available and In this case a well-defined pricing density exists. When dealing with new, incomplete markets with low liquidity options like carbon market it appears necessary to build new pricing approaches. The payoff Θ_t is defined as:

$$\begin{aligned}\Theta_t &= E_Q[\Theta_T e^{-r(T-t)} \mid \Phi_t] \\ &= e^{-r(T-t)} \int_0^\infty \Theta_T(S_T) q_{t,T}(S_T) dS_T \\ \Theta_t &= E_Q[\Theta_T(S_T) M_{t,T} \mid \Phi_t] \\ &= \int_0^\infty \Theta_T(S_T) M_{t,T} p_{t,T}(S_T) dS_T\end{aligned} \quad (1)$$

where t is the valuation moment, T is the option horizon, $q_{t,T}$ is the SDF (the PDF under the business (risk neutral) measure $\boldsymbol{Q}$, $p_{t,T}$ is the PDF under the physical measure $\boldsymbol{P}$ and $M_{t,T}$ the rotation operator, given for the classic Gaussian case by the Radon-Nicodym derivative.

Once the dynamic under the historical probability has been specified throughout statistical modellings, we may overcome the problem of option pricing in incomplete low liquid markets adopting one of the three following points of view:

- imposing some constraints on the form of the Stochastic Discount Factor (SDF) $q_{t,T}$ or on the rotation $M_{t,T}$,
- developing an empirical method that makes sense from an economic (business) point of view,
- choosing a particular measure that minimizes the variance of the hedging loss as presented in equation (2), Follmer-Schweiser (14)

$$\min(\Theta_T - c - \int_0^T \varphi_t dS_t) \quad (2)$$

where φ_t is the amount of economy S_t held for hedging at the moment t and c is the initial option bill.

For the pricing approach developed in this section we assume the option valuation distribution similar in shape to the distributions of the historical underlying data set (in our case GH and GARCH (APARCH)). GH processes have an interesting feature due to their capacity to capture the four moments of a physical economy and can replicate accurately series with pronounces skewness and kurtosis. The Generalized Hyperbolic (Eberlein and Prause (13) and Barndorff-Nielsen (5)) distributions are characterized by 5 parameters with a shape parameter which permits very specific profiles to be obtained.

Considering a discrete time economy with S_t the asset price at time t, the one period rate of return is supposed to be conditionally distributed under the probability measure $\boldsymbol{P}$ as follows:

$$Y_t = \ln \frac{S_t}{S_{t-1}} = r + \psi\sqrt{h_t} + \varepsilon_t \quad (3)$$

where ε_t has zero mean and conditional variance h_t under the measure $\boldsymbol{P}$; r is the one-period risk-free rate of return and ψ the constant unit risk premium. We further assume under the framework described by Bollerslev (4) that ε_t follows a GARCH(1,1) process under $\boldsymbol{P}$ as showed formally,

$$\begin{aligned}&\varepsilon_t \mid \phi_t \propto N(0, h_t) \\ &h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}\end{aligned} \quad (4)$$

where Φ_t is the corresponding σ-algebra generated by the previous and present information. To inure the covariance stationarity of the GARCH (1,1) we assume that $\alpha_1 + \beta_1 < 1$.

We extend the previous form in two directions. First we enrich the GARCH models with Generalized Hyperbolic innovations. Second we account the scaling leverage effect on volatility through the Asymmetric Power-ARCH (APARCH) model described by the seminal paper of Ding, Granger and Engle (10). Thus in the equation (4) ε_t we consider GH distributed under the measure $\boldsymbol{P}$ and a power factor ζ for the GARCH dependency.

$$\begin{aligned}&\varepsilon_t \mid \phi_t \propto GH(\lambda; \frac{\alpha}{\sqrt{h_t}}; \frac{\beta}{\sqrt{h_t}}; \mu\sqrt{h_t}; \delta\sqrt{h_t}) \\ &h_t^{\varsigma/2} = \alpha_0 + \alpha_1 \varepsilon_{t-1}^{\varsigma} + \beta_1 h_{t-1}^{\varsigma/2}\end{aligned} \quad (5)$$

We can notice that equation with ς=2 matches the classic GARCH model with GH innovations. As soon as the modelling under the historical measure is fixed, in order to propose a pricing framework adapted to the carbon allowances market we introduce a generalization of the concept of local risk neutral valuation relationship (LRNVR) introduced by Duan (11). We use an empirical approach developed by Duan and Simonato (12) and Guégan-Ielpo (7) for obtaining a martingale equivalent under the business measure, thereby bypassing the complex computation of the related SDF.

Following Duan (11), we first introduce the notion of local measure and its properties:

(i) A measure $\hat{P}_t$ defined at a moment t is said to be local a local physical measure (LPM) if $S_t/S_{t-1} \mid \Phi_{t-1}$

distributes lognormaly under $\hat{P}_t$ and has a volatility equal to the instantaneous volatility.

(ii) A family of pricing measures $\overline{Q}$ is said to satisfy the generalized local risk neutral valuation relationship (GLRNVR) if for any measure ***Q*** belonging to this family the two relations are satisfied

$$E_Q(S_t / S_{t-1} | \Phi_{t-1}) = e^r$$
$$Var_Q(\ln(S_t / S_{t-1}) | \Phi_{t-1}) = Var_{\hat{P}_t}(\ln(S_t / S_{t-1}) | \Phi_{t-1}) \quad (6)$$

almost surely with respect to the local physical measure $\hat{P}_t$.

Now to rule out arbitrage opportunities, we directly impose risk neutral constraints. The i^{th} sampled final price for the underlying is denoted by $S_{T,i} = S_t \prod_{k=t+1}^{T} e^{Y_k}$, where Y_k is given by (3). We consider the Empirical Martinagle Correction described by Chorro- Guegan (7) by the process that replaces the sampled prices $S_{T,i}$ with:

$$\overline{S}_{T,i} = \frac{S_{T,i}}{\frac{1}{N}\sum_1^N S_{T,i}} S_t e^{r(T-t)} \quad (7)$$

With this approach, we only shift the historical distribution in a way that prevents arbitrage opportunities by implicitly changing the drift of this distribution. The sampled average of $\overline{S}_{t,i}$ risk neutralized satisfies:

$$E_Q(\overline{S}_{t+1} / \overline{S}_t | \Phi_t) = e^r \quad (8)$$

Under this framework if σ_{P_t} represents the volatility of S_t under ***P*** and $\sigma_{\hat{P}_t}$ represents the instantaneous volatility under $\hat{P}_t$, then the ***Q***-economy defined as $\overline{Y^*_{t|Q}} = \ln(\overline{S^*_t} / \overline{S^*_{t-1}})$, $\overline{S^*_t}$ being the EMC re-sampled prices S^*_t with a return Y^*_t given by:

$$Y^*_t = \overline{Y}_t * \frac{\sigma_{\overline{P}_t}}{\sigma_P} \quad (9)$$

respects the generalized local risk neutral valuation relationship.

Applying (9) to the GH model we obtain the following form under ***Q:***

$$Y^*_t \propto GH(\lambda; \frac{\alpha}{\sqrt{h_t}}\frac{\sigma_{P_t}}{\sigma_{\overline{P}_t}}; \frac{\beta}{\sqrt{h_t}}\frac{\sigma_{P_t}}{\sigma_{\overline{P}_t}}; \eta^*_t + \mu\sqrt{h_t}\frac{\sigma_{\overline{P}_t}}{\sigma_P}; \delta\sqrt{h_t}\frac{\sigma_{\overline{P}_t}}{\sigma_P})$$

where η^*_t is mitigated via the Empirical Martingale Correction

III. ECONOMETRIC MODELLING

We calibrate several GH distributions using the full historical data set with various values for λ and we provide the results in Table 1. In order to compare the adequacy of the different modellings to the EUA historical prices on period 2005-2010, we use the Akaike Information Criteria (AIC) and the Bayesian Information Criteria (BIC) (in our writing, the lower the value of the BIC/AIC, the better fit we obtained). The results given in Table 1 show that GH models outperform both pure Gaussian and Gaussian Jump models. Amongst the various GH models the best results are provided for values λ =-0.5 of corresponding to the NIG distribution and λ =0.

Considering this extended framework of GARCH modelling described in the previous section we ***P***-estimated the parameters on our data-set fixing the values for ζ and λ. Thus we obtained parameterizations for APARCH models with various shapes of innovations and different level of leverage. In order to limit the perimeter of our possible results we choose the NIG as main innovation distribution. The calibration of APARCH processes for other values of λ give very close results. We choose for benchmark purpose the classic GARCH diffusion and the GARCH with t-Student innovations.

The results provided in Table 2 classified depending on BIC(AIC) score show that the non-Gaussian innovation GARCH models outperform Bollerslev's GARCH, APARCH models capture more information than the simple GARCH for equivalent innovations' form and APARCH with small ζ and NIG innovations provide with the best fitting results.

TABLE 1. PARAMETERS' ESTIMATION FOR GH MODELS

Model	AIC	BIC
Gaussian	-4292	-4282
Gaussian with Jumps	-4518	-4494
GH(λ=-0.5) NIG	-4560	-4540
GH (λ=-1)	-4558	-4538
GH (λ=-1.5)	-4552	-4532
GH (λ=0)	-4560	-4540
GH (λ=0.5)	-4558	-4538
GH (λ=1)	-4552	-4532

TABLE 2. PARAMETERS' ESTIMATION FOR GARCH (APARCH) MODELS

Model	AIC	BIC
GARCH Gaussian	-4494	-4479
APARCH (ζ=1.5) Gaussian	-4516	-4501
APARCH(ζ=1) Gaussian	-4544	-4529
GARCH t-Student	-4554	-4534
APARCH(ζ=2.5) NIG	-4724	-4700
GARCH NIG	-4732	-4708

APARCH(ζ=1.5) NIG	-4742	-4718
APARCH(ζ=0.5) NIG	-4776	-4752

IV. Option pricing

We implement now the Empirical Martingale Correction approach introduced previously. First we estimate with a Maximum Likelihood algorithm the distribution parameters under the physical measure ***P*,** using a sample of past year daily returns. Second we estimate the volatility under the local physical measure $\hat{P}_t$ using the implied Black and Scholes volatility given by the recent transactions dealt previously to the moment *t*. The use of the implied volatility as proxy for the local volatility is justified by a practitioner's view, the framework being open to other approximations. Third we simulation a Monte Carlo sample $S_{T,i}$ under ***P*** and we rescale the stationary variance as showed in the previous section. The fourth steps concern the drift neutralization with an Empirical Martingale Correction of the rescaled sample, thereby placing our simulation in a business perspective. Finally we price the option by discounting the payoffs generated by the previous sample.

The model makes three major assumptions that impact directly the pricing results. First it is considered that the shape distribution under ***P*** is the same under ***Q***. We believe that that business perspective is similar with the historical perspective. Various authors from the headstone work of Black (2) to more recent Barone-Adesi (1), and Chorro et al (7) discussed previously this issue, but we assume that in an illiquid and incomplete market, the option price incorporates a 'psychological premium' materializing the belief that the extreme variations could reproduce in the future.

TABLE 3. BREAKDOWN BY MATURITY AND MONEYNESS OF OUR OPTION SAMPLE

Moneyness/ Maturity	**T≤ 0.5**	**0.5<T≤ 1**	**1<T**
M ≤ 0.75	23	151	196
0.75<M ≤ 0.95	49	97	107
0.95<M ≤ 1.05	212	340	200
1.05<M ≤ 1.25	100	311	167
1.25<M	10	169	143

Thus the model capturing the asymmetry, the heavy tails and the volatility pattern would be also appropriate for pricing options. Second it is assumed that the skewness and kurtosis in the business measure are given by the physical calibration. We perceive the superior moments as a breach of the Efficient Market Hypothesis and as an intrinsic feature of the market regardless of the measure. Third the variance is rescaled to its local neutral value. The results are tested on carbon options cleared via ECX on the period 2008-2010, the sample accounting for 2275 transactions. The pricing is made via Monte Carlo simulation with 30 000 trajectories. We compare the average relative and absolute pricing errors across models, *K* is the strike and *T* the maturity:

$$RPE = \frac{1}{N}\sum_{T}\sum_{K}\left|\frac{C_t(T,K) - C^{estim}{}_t(T,K)}{C_t(T,K)}\right|$$
$$APE = \frac{1}{N}\sum_{T}\sum_{K}\left|C_t(T,K) - C^{estim}{}_t(T,K)\right| \quad (10)$$

Tables 4 and 5 show the interest to compare the Gaussian framework with a more flexible one which can be more accurate for certain maturities and moneyness. Our results show that the pure NIG provides with the lower pricing errors for a lot of moneyness and maturities. The short maturities options are comfortable with pricing based on Gaussian models. The previous table shows that for short maturities and very out-of-the-money of in-the-money options the GARCH models perform better. Thus we found by a different approach classic results that confirms the ability of GARCH to model to deal with moneyness issues (11). The NIG models offer better prices for long maturities options. Surprisingly the NIG seems to explain better the prices of short and medium-term at-the-money options, due mainly to the asymmetry effect. The Gaussian models perform for short-maturities out and in the money.

TABLE 4. PRICING ERRORS FOR VARIOUS CANDIDATE MODELS

Model	**RPE**	**APE**
Gaussian	0.156	0.277
NIG	**0.142**	**0.252**
GARCH	0.149	0.262
GARCH NIG	0.155	0.275
PARCH	0.151	0.261
PARCH NIG	0.156	0.270

TABLE 5. BEST PRICING MODEL BY MATURITY AND MONEYNESS OF OUR OPTION SAMPLE

Moneyness/ Maturity	**T≤ 0.5**	**0.5<T≤ 1**	**1<T**
M ≤ 0.75	GARCH	GARCH NIG	NIG
0.75<M ≤ 0.95	Gaussian	NIG	Gaussian
0.95<M ≤ 1.05	NIG	NIG	APARCH
1.05<M ≤ 1.25	Gaussian	NIG	NIG
1.25<M	GARCH	NIG	NIG

PARCH models bring less added pricing accuracy when estimated on short windows, but they are more effective on long-term maturities especially for at-the-money strikes

V. Conclusions

Compared to other underlying the emission permits are a very particular class with serious issues of information completeness and market efficiency. This work brings a practical pricing methodology, backed by a theoretical framework that enables the trader to valuate derivative in a market with low vanilla liquidity. In order to deal with the impossibility of multi-parameter model calibration we use a method confining some ideas brought by Duan (11) and (7). The major step forward of the present work is the variance rescaling that makes possible an accurate pricing in absence of observables data. One might be able to price a derivative underwritten on a non-Gaussian underling without any market proxies for similar transactions.

Future works will complete the present study with consideration of hedging effectiveness and the robustness of risk measure for our modeling.

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