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Disaggregation of spatial autoregressive processes

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***Abstract*—An aggregated Gaussian random field, possibly strong-dependent, is obtained from accumulation of i.i.d. short memory fields, via an unknown mixing density φ which is to be estimated. The so- called disaggregation problem is considered: i.e. φ is estimated from a sample of the limiting aggregated field while samples of the elementary processes remain unobserved. Estimation of the density is via its expansion in terms of orthogonal Gegenbauer polynomials. After defining the estimators, their asymptotic properties and convergence rates are discussed. The results hold either for long- and short-range dependent aggregated fields.**

Keywords-Gegenbauer polynomials; aggregated random fields; Gaussian random field; long-memory; density estimation.

I. INTRODUCTION

Consider the spatial autoregressive model

$$X_{s,t} = \alpha_1 X_{s-1,t} + \alpha_2 X_{s,t-1} - \alpha_1\alpha_2 X_{s-1,t-1} + \varepsilon_{s,t}, \quad (s,t)\in Z^2, \qquad (1)$$

where $-1 < \alpha_1,\alpha_2 < 1$ and $\varepsilon_{s,t}$, $(s,t)\in Z^2$ is a white noise with null mean and variance σ^2. This model was introduced by Martin (1979). If a random field is reflection-symmetric, that is $\rho_{k,l} = \rho_{-k,-l} = \rho_{k,-l} = \rho_{-k,l}$; then the autocorrelation function is of the form

$$\rho_{k,l} = \alpha_1^{|k|}\alpha_2^{|l|}, \quad (k,l)\in Z^2 .$$

Accumulation of short-memory random variables can lead to long-memory macro phenomena and provides an explanation for the appearance of strong memory in many applications, for further details see Granger (1980), Leonenko and Taufer (2005), Zaffaroni (2004) and the references therein. In this paper we consider the so-called disaggregation problem: having data from an aggregated field at hand, and assuming that these data are accumulated from i.i.d. short memory dynamics, the objective is to recover the distribution of the individual fields.

Accumulation is obtained through a sequence of independent random fields $X^{(j)}_{s,t}$, $(s,t)\in Z^2$, j=1,2 …, defined as,,

$$X^{(j)}_{s,t} = \alpha_1^{(j)} X^{(j)}_{s-1,t} + \alpha_2^{(j)} X^{(j)}_{s,t-1} - \alpha_1^{(j)}\alpha_2^{(j)} X^{(j)}_{s-1,t-1} + \varepsilon^{(j)}_{s,t}, \qquad (2)$$

where

I. $\varepsilon^{(j)}_{s,t}$, $j=1,2$ … is a sequence of independent copies of zero mean and unit variance strong white noise, i.e $\varepsilon^{(j)}_{s,t}$, $(s,t)\in Z^2$ are independent in (s,t);

II. $(\alpha_1^{(j)}, \alpha_2^{(j)})$, $j=1,2,$ … are independent copies (in j) of a random vector (α_1,α_2) supported in $[-1,1]\times[-1,1]$ and satisfying E $m(\alpha_1, \alpha_2) < \infty$ for some function $m(\alpha_1, \alpha_2)$.
In the simplest case α_1 and α_2 are independent and

$$E\frac{1}{1-\alpha_1^2} < \infty, \qquad E\frac{1}{1-\alpha_2^2} < \infty.$$

III. The sequences $\{\alpha_1^{(j)}, \alpha_2^{(j)}\}_{j\geq 1}$ and $\{\varepsilon^{(j)}\}_{j\geq 1}$ are independent.

Under I.-III. the finite dimensional distribution of the random fields weakly converge to those of a zero mean Gaussian field, i.e.

$$Y^{(N)}_{s,t} = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} X^{(j)}_{s,t} \to Y_{s,t}, \qquad (s,t)\in Z^2 \qquad (3)$$

in distribution as $N\to\infty$. The field $Y_{s,t}$, $(s,t)\in Z^2$ is called an aggregated. We suppose that the distribution of $(\alpha_1^{(j)}, \alpha_2^{(j)})$ admits a mixture density $\varphi(x_1,x_2)=\varphi_1(x_1)\varphi_2(x_2)$ (independent case) such that

$$\int_{-1}^{1}\int_{-1}^{1}\frac{\varphi_1(x_1)\varphi_2(x_2)}{(1-x_1^2)(1-x_2^2)}dx_1dx_2 < \infty. \qquad (4)$$

The aggregated field (3) has the same covariance function of the individual fields, i.e.,

$$\sigma(s,t) = Cov(X^{(j)}_{s,t}, X^{(j)}_{0,0}) = Cov(Y_{s,t}, Y_{0,0}) = \int_{-1}^{1}\int_{-1}^{1}\frac{|x_1|^s|x_2|^t}{(1-x_1^2)(1-x_2^2)}\varphi_1(x_1)\varphi_2(x_2)dx_1dx_2. \qquad (5)$$

The long range dependent case, that is $\sum_{(s,t)\in Z^2}|\sigma(s,t)| = \infty$, is obtained if and only if

$$\int_{-1}^{1}\int_{-1}^{1}\frac{\varphi_1(x_1)\varphi_2(x_2)}{(1-x_1^2)(1-x_2^2)}dx_1dx_2=\infty$$

The disaggregation problem deals with finding the individual fields (if they exist) of the form (2) which produce the aggregated field (3) with covariance (5). This is equivalent to finding $\varphi(x_1,x_2)$, the mixture density such that (5) holds, or to estimate $\varphi(x_1,x_2)$ using the aggregate observations $Y_{s,t}$, $1 \le s \le n$, $1 \le t \le n$.

II. Estimation

Let $G_m^{(\alpha)}(x)$, $x\in[-1,1]$, $m\in N$, denote the normalized Gegenbauer polynomials. These can be defined as $h_m^{-1/2} C_m^{\alpha+1/2}(x)$ where the quantities $h_m^{-1/2}$ and $C_m^{\alpha}(x)$ are defined in Abramowitz and Stegun (1965, formula 22.2.3).
Let $A_u=[-u,u]$ and consider the densities

$$w_{\alpha_j}(x_j)=(1-x_j^2)^{\alpha_j},\quad \alpha_j>-1,\quad x_j\in A_1,\quad j=1,2.$$

Define the expectation operator $E^W(.)$ as

$$E^W[h(X_1,X_2)]=\int_{A_1^2}h(x_1,x_2)w_{\alpha_1}(x_1)w_{\alpha_2}(x_2)dx_1dx_2.$$

The set of bivariate Gegenbauer orthogonal polynomials $\{G_{K_1,K_2}^{(\alpha_1,\alpha_2)}\}_{K_1,K_2\in N}$, where

$$G_{K_1,K_2}^{(\alpha_1,\alpha_2)}(x_1,x_2)=G_{K_1}^{(\alpha_1)}(x_1)G_{K_2}^{(\alpha_2)}(x_2),\qquad K_1,K_2\in\{0,1,...\}$$

forms a complete orthogonal system in the Hilbert space

$$L_2(A_1^2,W)=\{h: E^W[h^2(X_1,X_2)]<\infty.$$

A function $h\in L_2(A_1^2,W)$ admits an expansion in mean square convergent series of the following form

$$h(x_1,x_2)=\sum_{K_1=0}^{\infty}\sum_{K_2=0}^{\infty}h_{K_1,K_2}G_{K_1,K_2}^{(\alpha_1,\alpha_2)}(x_1,x_2),\tag{6}$$

with $h_{K_1,K_2}=E^W[h(X_1,X_2)G_{K_1,K_2}^{(\alpha_1,\alpha_2)}(X_1,X_2)]$.

Exploiting the ideas developed in Leipus et al. (2006) an estimator of the bivariate density $\varphi_1(x_1)\varphi_2(x_2)$ will be developed in terms of the first few terms of the orthogonal expansion (6). This is possible under the condition

$$\int_{-1}^{1}\int_{-1}^{1}\frac{\varphi_1^2(x_1)\varphi_2^2(x_2)}{(1-x_1^2)^{\alpha_1}(1-x_2^2)^{\alpha_2}}dx_1dx_2<\infty,\tag{7}$$

with $\alpha_j > -1$, $j=1,2$. The covariance $\sigma(j_1,j_2)$ can simply be estimated by the sample covariance

$$\hat\sigma_n(j_1,j_2)=\frac{1}{n^2}\sum_{i_1=1}^{n_1-j_1}\sum_{i_2=1}^{n_2-j_2}X_{i_1,i_2}X_{i_1+j_1,i_2+j_2}$$

and the natural estimate of the mixture density $\varphi_1(x_1)\varphi_2(x_2)$ is of the form

$$\hat\varphi(x_1,x_2)=(1-x_1^2)^{\alpha_1}(1-x_2^2)^{\alpha_2}\times\sum_{K_1=0}^{K_1(n)}\sum_{K_2=0}^{K_2(n)}\hat\zeta_{K_1,K_2}^{(n)}G_{K_1,K_2}^{(\alpha_1,\alpha_2)}(x_1,x_2)\tag{8}$$

where

$$\hat\zeta_{K_1,K_2}^{(n)}=\sum_{j_1=0}^{K_1}\sum_{j_2=0}^{K_2}g_{K_1,j_1}^{(\alpha_1)}g_{K_2,j_2}^{(\alpha_2)}[\hat\sigma(j_1,j_2)-\hat\sigma(j_1,j_2+2)+ -\hat\sigma(j_1+2,j_2)+\hat\sigma(j_1+2,j_2+2)]\tag{9}$$

and $K_1(n)$, $K_2(n)$ are non-decreasing sequences which tend to infinity with special rate.

The next theorem gives a consistency result for the estimator $\hat\varphi_n=\hat\varphi_n(x_1,x_2)$.

Theorem 1. Let $\{X_{s,t}\}$ be the aggregated field defined by (3) with a mixture density satisfying (4) $\alpha_j>-1$, $j=1,2$ are such that (7) holds. For $K_1(n)$ and $K_2(n)$ satisfying $K_j(n)=\gamma_j\log(n)$, $\gamma_j>0$, $j=1,2$, $\gamma_1+\gamma_2<(2\log(1+\sqrt{2})^{-1}$. Then $\hat\varphi_n$ is a consistent estimator of φ.

The proof relies on properties of Gegenbauer polynomials and their coefficients as well as the use of the diagram formula for cumulants.

Further results on L_2 and uniform convergence rates are obtained.

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