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A two-item deterministic EOQ model with partial backordering and substitution

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***Abstract*—There have been many inventory models with partial backordering, but few of them considered the case where the unmet demand can be satisfied by the substitutable item. In this paper, we propose a two-item deterministic EOQ model, where the demand of one item can be partially backordered and part of its lost sales can be satisfied for by the substitution. Our analysis provides a tractable and accurate method to determine order quantities and order cycles for the two items. The optimal solutions of the model, as well as the inventory decision procedures, are also developed.**

Keywords-Inventory;EOQ;Substitution;Partial backordering

I. INTRODUCTION

Generally, if stockouts happen, the unmet demand can be either backordered or lost. If the unsatisfied customers are willing to wait, then their demand can be met in the next replenishment epoch. Otherwise, they may buy their desired items from other suppliers, and the unmet demand is therefore lost immediately. In practice, the most frequent case is that some of the unsatisfied customers may be willing to wait and backorder their unmet demands while some others may purchase their desired items from another vendor, which represents the case of partial backordering. Montgomery et al. (1973) presented the first model on EOQ with partial backordering, assuming that a fixed fraction of demand during the stock-out period is backordered, and the remaining fraction is lost. After that, Rosenberg (1979), Park (1982), and Pentico and Drake (2009a) proposed some similar EOQ models with partial backordering.

If there are some similar items in stock, substitutions may occur when one of the demanded items is stocked out. During all the stocked out period, the unsatisfied customers may chose another similar item, which forms an inventory problem of the EOQ with substitution. The earliest literature refers to the substitution of products is Veinott (1969). McGillivray and Silver (1978) presented the concepts of substitutable items in inventory management. They assumed that a proportion of the unmet demand can be satisfied by another similar item. After that, many studies on inventory models with substitution were proposed (Parlar and Goyal, 1984; Parlar, 1985; Drezner et al., 1995; Ernst and Kouvelis, 1999; Rajaram and Tang, 2001; Netessine and Rudi, 2003; Nagarajan and Rajagopalan, 2008; Huang et al, 2010).

Most inventory models with substitution assume that partial demand of stocked out item can be satisfied by substitutable item, and the rest of the unmet demand is lost. This assumption may be realistic premising that the stocked out item can not be backordered. However, it is frequent in practice that some of unsatisfied customers are willing to select substitutable item, some are willing to backorder their first-choice item and some others will choose another vendor leading to lost sales. Therefore, it is more realistic to consider three factors in inventory model: substitution, partial backordering and lost sales.

In this paper, the authors develop a two-item EOQ model with partial backordering in consideration of substitutable item. In the proposed model, one of the two items called major item, the demand of which is independent and can be partially backordered with lost sales. The other one is the substitutable item whose demand will increase because of the substitution during the period when the major item is stocked out. Since the demands of the two items are correlated, a new joint inventory policy should be pursued to optimize the inventory management.

The rest of the paper is organized as follows. In section 2, we propose the two-item EOQ model with partial backordering and substitution, and the optimal solution is derived. Section 3 presents the decision procedures for yielding the optimal inventory policy. Section 4 carries out the computational study for illustration, comparison and performance analysis. In the last section we provide some conclusions.

II. MODELING AND SOLUTION

A. Assumptions and notations

Considering the scenario where a major item is stocked out, partial customers are willing to backorder the unsatisfied demand. The rest are willing to select the new innovative item as substitution or select a new vendor leading to a lost sale. A joint replenishment inventory policy should be pursued, for which the following assumptions are made.

1. The demand rates of the two items are constant.
2. Replenishment is instantaneous.
3. The lead time is zero.
4. The substitutable item should not be stocked out.
5. The substitution rate α and the backordering rate β are constant, which are larger than 0 and less than 1 ($0 < \alpha < 1, 0 < \beta < 1$).

The inventory levels of the two items over the course of an order cycle and their relationships to demand are shown in Fig. 1.

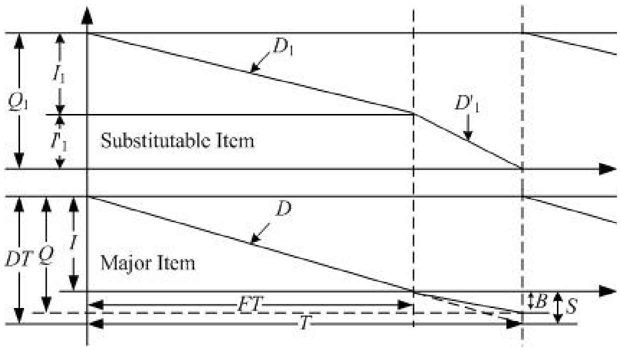

Figure 1. The inventory levels of the two items with partial backordering and substitution.

Parameters

D—demand per unit time of the major item, in units/unit time

A—the fixed cost of placing and receiving an order for the major item, in dollars

C_h—the cost to hold one unit item in inventory for one unit time for the major item, in \$/unit/unit time

C_b—the cost to keep one unit item backordered for one unit time for the major item, in \$/unit/unit time

C_o—the opportunity cost of one unit lost sale for the major item, including the lost profit and goodwill loss, in \$/unit

α—the fraction of the major item stockouts that will be substituted

β—the fraction of the major item stockouts that will be backordered

D_1—demand per unit time of the substitutable item, in units/unit time

A_1—the fixed cost of placing and receiving an order for the substitutable item, in dollars

C_{h1}—the cost to hold one unit item in inventory for a year for substitutable item, in \$/unit/unit time

C_s—the cost to substitute one unit of the substitutable item for the major item (including the profit differential between the items), in \$/unit

Variables

T—the order cycle, i.e. the time interval between two replenishments

F—the fill rate, i.e. the percentage of demand that is filled from the shelf stock

B. The model with partial backordering ($F \in [0,1)$)

Considering F<1, i.e. the inventory policy is to meet demand with partial backordering, the total inventory cost of the two items can be represented as

$$\Gamma(T,F)=\frac{A+A_1}{T}+\frac{(C_h-\alpha C_{h1}+\beta C_b)DTF^2}{2}-\beta C_b DTF+ \\ [(1-\beta)C_o+\alpha C_s]D(1-F)+\frac{C_{h1}(D_1+\alpha D)+\beta C_b D}{2}T \quad (1)$$

Let $d=D_1/D$. Taking the first partial derivatives of $\Gamma(T,F)$, with respect to T and F respectively and setting them equal to 0 gives

$$T^*=\sqrt{\frac{2(A+A_1)}{D}\cdot\frac{C_h-\alpha C_{h1}+\beta C_b}{(d+\alpha)(C_h-\alpha C_{h1})C_{h1}+\beta(C_h+dC_{h1})C_b}-\frac{[(1-\beta)C_o+\alpha C_s]^2}{(d+\alpha)(C_h-\alpha C_{h1})C_{h1}+\beta(C_h+dC_{h1})C_b}} \quad (2)$$

$$F^*=\frac{\beta C_b T^*+(1-\beta)C_o+\alpha C_s}{(C_h-\alpha C_{h1}+\beta C_b)T^*} \quad (3)$$

Correspondingly, the order quantities of the two items can be given by Eq. (4).

$$Q^*=DF^*T^*+\beta D(1-F^*)T^* \\ Q_1^*=[D_1+\alpha D(1-F^*)]T^* \quad (4)$$

III. The Optimal Policy

The pair (T*, F*) given by Eqs. (2) and (3) may be a global minimum, a local minimum or only a stationary point of the function $\Gamma(T,F)$. For brevity in mathematics, denoting

$$g_0=A+A_1,$$

$$g_1=\frac{(C_h-\alpha C_{h1}+\beta C_b)D}{2},$$

$$g_2=\frac{\beta C_b D}{2},$$

$$g_3=[(1-\beta)C_o+\alpha C_s]D,$$

$$g_4=\frac{C_{h1}(D_1+\alpha D)+\beta C_b D}{2},$$

the cost function can be rewritten as $\Gamma(T,F)=\frac{g_0}{T}+u(F)T+v(F)$ where $u(F)=g_1F^2-2g_2F+g_4$ and $v(F)=g_3(1-F)$. By derivatives, we can prove the following property for the optimal solution.

A. The case of $C_h-\alpha C_{h1}>0$

If $C_h-\alpha C_{h1}>0$, it is easy to prove that $\frac{d^2\overline{\Gamma}(F)}{dF^2}>0$ holds for all F, which demonstrates that the function $\overline{\Gamma}(F)$ is convex in F. Note $\left.\frac{\partial\overline{\Gamma}}{\partial F}\right|_{F=0}=\frac{-2g_2}{\sqrt{u(F)}}-g_3<0$ so that all possible shapes of the function $\overline{\Gamma}(F)$ are pictured in Fig. 2.

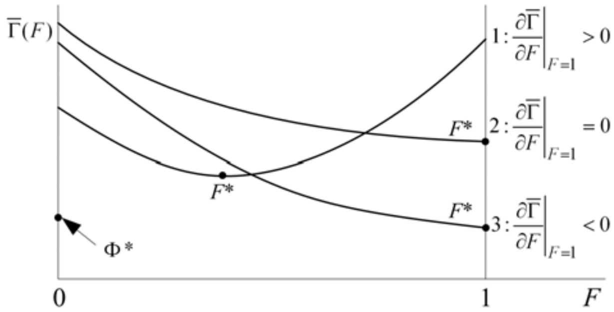

Figure 2. All possible shapes of the function $\overline{\Gamma}(F)$

As shown in Fig. 2, only if $\left.\frac{\partial\overline{\Gamma}}{\partial F}\right|_{F=1} > 0$, can the minimum point of $\overline{\Gamma}(F)$, i.e. F*, be attained on $F \in (0,1)$. In this case, we have $\left.\frac{\partial\overline{\Gamma}}{\partial F}\right|_{F=1} = \sqrt{g_0}\frac{2g_1 - 2g_2}{\sqrt{g_1 - 2g_2 + g_4}} - g_3 > 0$, i.e.,

$$\beta > \beta_s^* = 1 + \frac{\alpha C_s}{C_o} - \frac{C_h - \alpha C_{h1}}{C_o} \cdot \sqrt{\frac{2(A + A_1)}{C_h D + C_{h1} D_1}} \tag{5}$$

Where β_s^* is the critical value of the backordering rate above which if $C_h - \alpha C_{h1} > 0$, the minimum of the total cost function $\Gamma(T, F)$ is attained on $F \in (0,1)$. According to Zhang (2009), in this case we should compare the cost of EOQ with partial backordering, i.e. $\Gamma(T^*, F^*)$ with the cost of forfeiting all demand, i.e. $\Phi^* = (C_o + \alpha C_s)D + \sqrt{2A_1 C_{h1}(D_1 + \alpha D)}$ to determine the optimal choice.

If $\beta \le \beta_s^*$, the minimum of $\overline{\Gamma}(F)$ is $F^*=1$ and the optimal inventory cost is determined by the basic EOQ without stockouts, i.e. $H^* = \sqrt{2(A + A_1)(C_h D + C_{h1} D_1)}$. We should compare with to determine the optimal policy (Zhang, 2009).

B. The case of $C_h - \alpha C_{h1} \le 0$

If $C_h - \alpha C_{h1} \le 0$, then $\left.\frac{\partial\overline{\Gamma}}{\partial F}\right|_{F=0} = \frac{-2g_2}{\sqrt{u(F)}} - g_3 < 0$ and $\left.\frac{\partial\overline{\Gamma}}{\partial F}\right|_{F=1} = \sqrt{g_0}\frac{2g_1 - 2g_2}{\sqrt{u(F)}} - g_3 < 0$, i.e., the function $\overline{\Gamma}(F)$ is therefore monotonously decreasing on the interval of $F \in [0,1]$. Thus, the optimal cost must be attained on $F=1$, i.e. to meet all demand of the major item without stockouts. Similar to the case of $C_h - \alpha C_{h1} > 0$, we should further compare the cost of the basic EOQ without stockouts , with the cost of forfeiting all demand to determine the optimal policy.

As a summary, the procedure for determining the optimal inventory policy is as follows.

1) Calculate $C_h - \alpha C_{h1}$

2) If $C_h - \alpha C_{h1} \le 0$, compare the cost of H^* with Φ^*.

a) If $H^* \ge \Phi^*$, then forfeit all demand of the major item.

b) If $H^* < \Phi^*$, then meet all demand without stockout.

3) If $C_h - \alpha C_{h1} > 0$, determine from Eq. (5).

a) If $\beta \le \beta_s^*$, compare H^* with Φ^*. The solution of the smaller one is the optimal policy.

b) If $\beta > \beta_s^*$, compare Γ^* with Φ^*. The solution of the smaller one is the optimal policy.

IV. NUMERICAL EXAMPLES FOR ILLUSTRATION

We use a numerical example to illustrate the application of the decision procedure given above. Consider problems for which the parameters are:

D=2000 units/year, A=\$500/order, C_h=\$20/unit/year, C_b=\$5/unit/year, C_o=\$10/unit/year

D_1=1500 units/year, A_1=\$400/order, C_{h1}=\$5/unit/year, α=0.2.

$C_h - \alpha C_{h1} = 20 - 0.2 \cdot 5 = 19 > 0$

Determine β_s^*:

First, we get

$$T_H^* = \sqrt{\frac{2(A + A_1)}{C_h D + C_{h1} D_1}} = \sqrt{\frac{2 \cdot (500 + 400)}{20 \cdot 2000 + 5 \cdot 1500}} = 0.1947$$

$$H^* = \sqrt{2(A + A_1)(C_h D + C_{h1} D_1)}$$

$$= \sqrt{2 \cdot (500 + 400) \cdot (20 \cdot 2000 + 5 \cdot 1500)} = 924662$$

β_s^* given by Eq. (5) is

$$\beta_s^* = 1 + \frac{\alpha C_s}{C_o} - \frac{C_h - \alpha C_{h1}}{C_o} T_H^*$$

$$= 1 + \frac{0.2 \cdot 2}{10} - \frac{20 - 0.2 \cdot 5}{10} \cdot 0.1947 = 0.6700$$

If $\beta = 0.6$, then $\beta \le \beta_s^*$, and the optimal policy is to either forfeit all demand of the major item or meet demand without stockouts. The cost of forfeiting all demand of the major item is

$$\Phi^* = (C_o + \alpha C_s)D + \sqrt{2A_1 C_{h1}(D_1 + \alpha D)}$$

$$= (10 + 0.2 \cdot 2) \cdot 2000 + \sqrt{2 \cdot 400 \cdot 5 \cdot (1500 + 0.2 \cdot 2000)} = 23556.81$$

Obviously, $H^* < \Phi^*$, so the optimal policy is to meet all demand without stockouts.

If $\beta = 0.7$, then $\beta > \beta_s^*$, and the inventory policy should be to either meet demand of the major item with partial backordering or forfeit all demand. If the policy is to meet demand with partial backordering, the optimal order cycle and the optimal fill rate given by Eqs. (2) and (3) are

$$T^* = \sqrt{\frac{2(A+A_1)}{D}\cdot\frac{C_h-\alpha C_{h1}+\beta C_b}{(d+\alpha)(C_h-\alpha C_{h1})C_{h1}+\beta(C_h+dC_{h1})C_b} - \frac{[(1-\beta)C_o+\alpha C_s]^2}{(d+\alpha)(C_h-\alpha C_{h1})C_{h1}+\beta(C_h+dC_{h1})C_b}}$$

$$= \sqrt{\frac{2\cdot(500+400)}{2000}\cdot\frac{20-0.2\cdot5+0.7\cdot5}{(0.75+0.2)\cdot(20-0.2\cdot5)\cdot5+0.7\cdot(20+0.75\cdot5)\cdot5} - \frac{[(1-0.7)\cdot10+0.2\cdot2]^2}{(0.75+0.2)\cdot(20-0.2\cdot5)\cdot5+0.7\cdot(20+0.75\cdot5)\cdot5}}$$

$$= 0.2239$$

$$F^* = \frac{\beta C_b T^* + (1-\beta)C_o + \alpha C_s}{(C_h-\alpha C_{h1}+\beta C_b)T^*}$$

$$= \frac{0.7\cdot5\cdot0.2239+10\cdot(1-0.7)+0.2\cdot2}{(20-0.2\cdot5+0.7\cdot5)\cdot0.2239} = 0.8305$$

Substituting T^* and F^* into Eq. (1), the inventory cost can be calculated as $\Gamma^* = 2\sqrt{g_0 u(F^*)} + v(F^*) = 9192.47$

The cost of forfeiting all demand is

$$\Phi^* = 23556.81 > \Gamma^* = 9192.47$$

Therefore, the optimal policy is to meet demand of the major item with partial backordering.

$I^* = DF^*T^* = 371.87 \quad S^* = (1-F)DT^* = 75.87$

$B^* = \beta S^* = 53.121 \quad Q^* = I^* + B^* = 425$

V. CONCLUSIONS

In this paper, assuming the lost sales of the major item can be partially satisfied through substitutable item, we propose a partial backordering deterministic EOQ model with substitutions. We prove that there exits a critical value of the backordering rate that determines whether the optimal policy is to meet demand with partial backordering, and this phenomenon is similar to the case of the single-item EOQ with partial backordering.

One of the possible extensions of this work is to take stochastic demand into account when building the EOQ model with partial backordering and partial substitution. Another extension is to consider multiple items in our model, where a more complicated model must be solved.

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