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Modeling the Risk by Credibility Theory

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Abstract. This paper is an attempt to treat some topics of risk theory by means of credibility theory. The risk aversion of an agent faced with a situation of uncertainty represented by a discrete fuzzy variable and the relationship between stochastic dominance and credibilistic dominance is under study.

Keywords: credibility theory, risk aversion, credibilistic dominance, credibilistic index of riskiness

1. Introduction

The modeling of uncertainty is traditionally done by probability theory [4], [12]. Zadeh's possibility theory [19] and Liu and Liu's credibility theory [13], [14] offers an alternative for modeling uncertainty.

The transitions from probabilistic models to possibilistic and credibilistic models require that random variables are replaced by fuzzy variables, and that probabilistic indicators (expected value, variance, etc.) are replaced with appropriate possibilistic or credibilistic indicators.

For the probabilistic approach two risk two main directions can be distinguished ([17], [12]): (a) How much riskier is a situation and how can one evaluate if a situation is riskier than another? (b) How the evaluation is done if an agent is more risk averse than another? Arrow [1], [2] and Pratt [16] contributed to (b) by creating the theory of risk aversion (see also [4], [12]). With respect to the issue (a) we recall the results of [11], [17] on stochastic dominance. A possibilistic approach to risk aversion is found in papers [8], [9].

This paper attempts to study the risk by credibility theory focusing on risk situations described by fuzzy variables. Our investigations are based a procedure of [13], [14] by which a random variable X_ζ is associated with a discrete fuzzy variable ζ. The expected value of X_ζ coincides with the credibilistic expected value of ζ.

Next, in Section 2 some basic notions (of possibilistic measure, possibilistic distribution, and credibilistic measure) and their relations are recalled. Section 3 deals with the membership function of a fuzzy variable w.r.t. a credibility measure and the credibilistic expected value $Q(\zeta)$ of a fuzzy variable ζ. Discrete random variable X_ζ is associated with a discrete fuzzy variable ζ so that $Q(\zeta)$ coincides with the probabilistic mean value $M(X_\zeta)$ ([13], [14]). In Section 4 the behavior of the operator $\zeta \longmapsto X_\zeta$ w.r.t. a utility function u is studied. The notion of credibility expected utility $Q(u(\zeta))$ of a discrete fuzzy variable ζ w.r.t. a utility function u is introduced. $Q(u(\zeta))$ coincides with the probabilistic expected value $M(u(X_\zeta))$, if u is strictly increasing. Section 5 tackles credibilistic risk aversion. The notion of credibilistic risk premium associated with ζ and a u is introduced. Using the operator $\zeta \longmapsto X_\zeta$ an approximate calculation formula for the credibilistic risk premium in case of discrete fuzzy variable is obtained. In Section 6 the credibilistic dominance is turned into stochastic dominance by the operator $\zeta \longmapsto X_\zeta$. The paper is concluded in Section 7.

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2. Possibility and Credibility

Notions and results on possibility and credibility measures, by [3], [5], [6], [13], [14], are recalled next.

Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ its powerset. The elements of Ω are interpreted as states and the elements of $\mathcal{P}(\Omega)$ as elements. For any $D \in \mathcal{P}(\Omega)$ we denote $D^c = \Omega - D$.

Definition 2.1 [6] A *possibility measure* on Ω is a function $\Pi: \mathcal{P}(\Omega) \to [0,1]$ such that:

(Pos 1) $P(\emptyset) = 0$; $P(\Omega) = 1$;

(Pos 2) $\Pi(\bigcup_{i\in I} D_i) = sup_{i\in I}\Pi(D_i)$, for any family $(D_i)_{i\in I}$, of subsets of Ω.

A *possibility distribution* on Ω is a function $\mu: \Omega \to [0,1]$ such that $sup_{x\in\Omega}\mu(x) = 1$; μ is said to be *normalized* if $\mu(x) = 1$ for some $x \in \Omega$.

Let Π be a possibility measure on Ω and μ a possibility distribution on Ω. Let us consider the functions $\mu_\Pi: \Omega \to [0,1]$ and $Pos_\mu: \mathcal{P}(\Omega) \to [0,1]$ defined by

$$\mu_\Pi(x) = \Pi(\{x\}), \text{ for any } x \in \Omega; \tag{1}$$

$$Pos_\mu(D) = sup_{x\in D}\mu(x), \text{ for any } D \subseteq \Omega \tag{2}$$

Proposition 2.2 [6] (i) μ_Π is a possibility distribution on Ω and Pos_μ is a possibility measure on Ω; and (ii) there exists a bijective correspondence between them: $\mu_{Pos_\mu} = \mu$ and $Pos_{\mu_\Pi} = \Pi$.

Definition 2.3 [14] A *credibility measure* on Ω is a function $Cr: \mathcal{P}(\Omega) \to [0,1]$ such that:

(Cred 1) $Cr(\Omega) = 1$;

(Cred 2) If $A, B \in \mathcal{P}(\Omega)$ then $A \subseteq B$ implies $Cr(A) \le Cr(B)$;

(Cred 3) For any $A \in \mathcal{P}(\Omega)$, $Cr(A) + Cr(A^C) = 1$;

(Cred 4) For any family $(A_i)_{i\in I}$ of subsets of Ω such that $supCr(A_i) < 1/2$ the following equality holds: $Cr(\bigcup_{i\in I} A_i) = sup_{i\in I}Cr(A_i)$.

If $a, b \in \mathbb{R}$ then we denote $a \vee b = \sup(a, b)$ and $a \wedge b = \inf(a, b)$.

Let Π be a possibility measure on Ω and Cr a credibility measure on Ω. Let us consider the functions $\Pi_{Cr}: \mathcal{P}(\Omega) \to [0,1]$, and $Cr_\Pi: \mathcal{P}(\Omega) \to [0,1]$, defined by

$$\Pi_{Cr}(A) = 2Cr(A) \wedge 1, \text{ for any } A \subseteq \Omega; \tag{3}$$

$$Cr_\Pi(A) = 1/2\,[\Pi(A) + 1 - \Pi(A^C)], \text{ for any } A \subseteq \Omega. \tag{4}$$

Proposition 2.4 (i) Π_{Cr} is a possibility measure on Ω and Cr_Π is a credibility measure on Ω; and (ii) there exists a bijective correspondence between them: $\Pi_{Cr_\Pi} = \Pi$ and $Cr_{\Pi_{Cr}} = Cr$.

Let $\mu: \Omega \to [0,1]$ be a possibility distribution and Pos_μ the possibility measure defined by (2). We denote by $Cr_\mu$the credibility measure associated with Pos_μ according to (4):

$$Cr_\mu(A) = 1/2\,[sup_{x\in A}\mu(x) + 1 - sup_{x\notin A}\mu(x)]. \tag{5}$$

3. Fuzzy Variables and Credibilistic Indicators

In this section we will present some credibilistic indicators associated with fuzzy variables (the credibilistic expected value and the credibilistic variance). From now on in this paper, we will assume that the set of states is the set $\mathbb{R}$ of real numbers. A *fuzzy variable* is an arbitrary function $\zeta: \mathbb{R} \to \mathbb{R}$.

Let $Cr: \mathcal{P}(\Omega) \to [0,1]$ be a credibility measure and ζ a fuzzy variable. We consider $\mu: \mathbb{R} \to [0,1]$, which is the membership function of ζ w.r.t. *Cr,* defined by $\mu(x) = 2Cr(\zeta = x) \wedge 1$ for any $x \in \mathbb{R}$.

Proposition 3.1 [14] (i) μ is a possibility distribution; and (ii) $Cr = Cr_\mu$.

Proposition 3.2 [14] if $A \subseteq \mathbb{R}$ then $Cr(\zeta \in A) = 1/2\,[sup_{x\in A}\mu(x) + 1 - sup_{x\notin A}\mu(x)]$.

Corollary 3.3 For any $r \in \mathbb{R}$ we have: (a) $Cr(\zeta \le r) = 1/2\,[sup_{x\le r}\mu(x) + 1 - sup_{x>r}\mu(x)]$; and (b) $Cr(\zeta > r) = 1/2\,[sup_{x>r}\mu(x) + 1 - sup_{x\le r}\mu(x)]$.

Let *Cr* be a credibility measure, $\zeta: \mathbb{R} \to \mathbb{R}$ a fuzzy variable, and μ the membership function of ζ w.r.t. *Cr*.

Definition 3.4 [13] The *credibilistic expected value* $Q(\zeta)$ of ζ is defined (assuming finite integrals) by $Q(\zeta) = \int_0^\infty \mathrm{Cr}(\zeta \geq \mathrm{r})\mathrm{dr} - \int_{-\infty}^0 \mathrm{Cr}(\zeta \leq \mathrm{r})\mathrm{dr}$.

Definition 3.5 [14] Assume that the credibilistic expected value $Q(\zeta) = \mathrm{e}$ exists. Then the credibilistic variance $Var(\zeta)$ of ζ is defined by $Var(\zeta) = \mathrm{Q}[(\zeta - \mathrm{e})^2]$.

Proposition 3.6 [14] If $a, b \in \mathbb{R}$, then $Q(\mathrm{a}\zeta + \mathrm{b}) = \mathrm{a}Q(\zeta) + \mathrm{b}$.

A fuzzy variable ζ is *discrete* if it takes a finite number of values. Let $\mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_\mathrm{n}$ be distinct the values of a discrete fuzzy variable ζ. Then, the membership function μ of ζ has the form:

$$\mu(x) = \begin{cases} \mu_1 & if \quad x = a_1 \\ \mu_2 & if \quad x = a_2 \\ \cdots & \\ \mu_n & if \quad x = a_n \end{cases} \tag{6}$$

We write it in the table, which is said to represent the discrete fuzzy variable ζ:

$$\mu: \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ \mu_1 & \mu_2 & \cdots & \mu_n \end{bmatrix} \tag{7}$$

Let us consider the following real numbers:

$$p_i = {}^1/_2\,[\max_{1 \leq j \leq n}\{\mu_j | a_j \leq a_i\} - \max_{1 \leq j \leq n}\{\mu_j | a_j < a_i\}] \tag{8}$$
$$+ {}^1/_2\,[\max_{1 \leq j \leq n}\{\mu_j | a_j \geq a_i\} - \max_{1 \leq j \leq n}\{\mu_j | a_j > a_i\}].$$

Proposition 3.7 [13] The real numbers $p_1, p_2, \dots, p_n$ verify the following conditions: (i) $p_i \geq 0,\ i = 1, \dots, n$; and (ii) $\sum_1^n p_i = 1$.

Then one can consider the following discrete random variable:

$$X_\zeta : \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ p_1 & p_2 & \cdots & p_n \end{bmatrix}. \tag{9}$$

Proposition 3.8 [13] The credibilistic expected value $Q(\zeta)$ of ζ coincides with the probabilistic expected value $M(X_\zeta)$ of X_ζ.

Remark 3.9 [13] If $\mathrm{a}_1 < \mathrm{a}_2 < \cdots < \mathrm{a}_\mathrm{n}$ then the probabilities $p_1, \dots, p_n$ get the form

$p_i = {}^1/_2\,[\bigvee_{j=1}^{i} \mu_j - \bigvee_{j=0}^{i-1} \mu_j] + {}^1/_2\,[\bigvee_{j=1}^{n} \mu_j - \bigvee_{j=i+1}^{n+1} \mu_j],\ i = 1, \dots, n$, where $\mu_0 = \mu_{n+1} = 0$.

4. The Operator $\zeta \longmapsto X_\zeta$

In the previous sections we associated a discrete random variable X_ζ with a discrete fuzzy variable ζ. This way we define an operator from the set of discrete fuzzy variables to the set of discrete random variables. In this section we will study the way the operator $\zeta \longmapsto X_\zeta$ behaves w.r.t. the expected utility.

A *utility function* is a function $u: \mathbb{R} \to \mathbb{R}$ of class C^2, strictly concave and strictly increasing.

Let $(\Omega, \mathcal{K}, P)$ be a probability space and $X: \Omega \to \mathbb{R}$ a random variable. If *u* is a utility function, then the function $u(X): u \cdot X: \Omega \to \mathbb{R}$ is a random variable.

The probabilistic expected value $M(u(X))$ is called *probabilistic expected utility* of *X* w.r.t. *u*.

Let $Cr: \mathcal{P}(\mathbb{R}) \to [0,1]$ be a credibility measure, ζ a fuzzy variable, and μ the membership function of ζ. If *u* is a utility function, then $u(\zeta) = u \circ \zeta: \Omega \to \mathbb{R}$ is a fuzzy variable. The credibilistic mean value $Q(u(\zeta))$ of $u(\zeta)$ is called *credibility expected utility* of ζ w.r.t. *u*.

We consider now the discrete fuzzy variable ζ and the associated random variable X_ζ given by tables:

$$\zeta : \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ \mu_1 & \mu_2 & \cdots & \mu_n \end{bmatrix};\ X_\zeta : \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ p_1 & p_2 & \cdots & p_n \end{bmatrix}. \tag{10}$$

Proposition 4.1 The discrete fuzzy variable $u(\zeta)$ and the discrete random variable $u(X_\zeta)$ are written as:

$$\mathrm{u}(\zeta): \begin{bmatrix} u(a_1) & u(a_2) & \cdots & u(a_n) \\ \mu_1 & \mu_2 & \cdots & \mu_n \end{bmatrix};\ u(X_\zeta): \begin{bmatrix} u(a_1) & u(a_2) & \cdots & u(a_n) \\ p_1 & p_2 & \cdots & p_n \end{bmatrix}. \tag{11}$$

Remark 4.2 By Proposition 4.1 (see proof in [10]), we have $u(X_\zeta) = X_{u(\zeta)}$.

Proposition 4.3 (see proof in [10]) $Q(u(\zeta)) = M(u(X_\zeta))$.

5. Credibilistic Risk Aversion

The notion of credibilistic risk premium associated with a fuzzy variable and a utility function is introduced next using the operator $\zeta \longmapsto X_\zeta$. We recall first the notion of probabilistic risk. Let *X* be a random variable and *u* a utility function. We assume that *u* has the class C^2, strictly concave and strictly increasing.

Definition 5.1 [1], [16] The *probabilistic risk premium* $\Pi = \Pi(X, u)$ is the unique solution of:

$$M(u(X)) = u(M(X) - \Pi). \tag{12}$$

Proposition 5.2 [1], [6] An approximate solution of equation (12) is given by

$$\Pi \approx -\frac{1}{2} Var(X) \frac{u''(M(X))}{u'(M(X))}. \tag{13}$$

Now, let $Cr: \mathcal{P}(\mathbb{R}) \to [0,1]$ be a credibility measure, ζ a fuzzy variable, and *u* a utility function.

Definition 5.3 The credibilistic risk premium $\lambda = \lambda(\zeta, u)$ associated with ζ and *u* is the solution of:

$$Q(u(\zeta)) = u(Q(\zeta) - \lambda), \text{ which is unique.} \tag{14}$$

An approximate formula for λ analogous to (13) is unknown, but the following result can be proved.

Proposition 5.4 An approximate solution of equation (14) is (see proof in [10]):

$$\lambda = \frac{u(Q(\zeta)) - Q(u(\zeta))}{Q(\zeta) u'(Q(\zeta))}. \tag{15}$$

Let *u* be a utility function, ζ be a fuzzy variable and X_ζ the associated random variable from tables (10).

Proposition 5.5 The credibilistic risk premium $\lambda = \lambda(\zeta, u)$ associated with ζ and *u* coincides with the probabilistic risk premium $\Pi = \Pi(X_\zeta, u)$ associated with X_ζ and *u* (see proof in [10]).

The previous proposition, the evaluation of risk aversion of an agent in front of a risk situation represented by a discrete fuzzy variable reduces to the evaluation of probabilistic risk aversion corresponding to the random variable X_ζ. The thesis is sustained by the evaluation formula of the following proposition.

Proposition 5.6 We consider a discrete fuzzy variable ζ and the discrete random variable X_ζ from tables (10). Then, an approximate value of credibilistic risk aversion $\lambda(\zeta, u)$ is (see proof in [10]):

$$\lambda(\zeta, u) = -\frac{1}{2} \frac{u''(Q(\zeta))}{u'(Q(\zeta))} \sum_{i=1}^{n} (a_i - Q(\zeta))^2 p_i. \tag{16}$$

6. Credibilistic and Stochastic Dominance

Credibilistic dominance was introduced in [11], [17] as a way of ranking the fuzzy variables. It is similar to stochastic dominance, a notion studied in probabilistic models of risk in [4], [7], [11], [12], [17]. In this section we will establish a relation between the credibilisitic dominance of discrete fuzzy variables and the stochastic dominance of discrete random variables associated with them by the operator $\zeta \longmapsto X_\zeta$.

We recall from [7] the notion of *stochastic dominance* of order *k*. Let $(\Omega, \mathcal{K}, P)$ be a probability space with $\Omega \subseteq \mathbb{R}$ and $X: \Omega \to \mathbb{R}$ a random variable. Then the distribution function $F_X: \mathbb{R} \to [0,1]$ associated with *X*: $F_X(x) = P(X \le x)$ for any $x \in \mathbb{R}$. For any $x \in \mathbb{R}$, we will define by induction

$$F_X^{[0]}(x) = F_X(x);\ F_X^{[k+1]}(x) = \int_{-\infty}^{x} F_X^{[k]}(t)dt. \tag{17}$$

Definition 6.1 Let *X and Y* be two random variables and $k \in \mathbb{N}$. We define $X \succcurlyeq_{prob}^{(k)} Y$ iff $F_X^{[k]}(x) \le F_Y^{[k]}(x)$, for any $x \in \mathbb{R}$. If $X \succcurlyeq_{prob}^{(k)} Y$, then we say that *X dominates stochastically Y in order k.*

Following [15], [20], the *credibilistic dominance* of order *k* is presented. Let $Cr: \mathcal{P}(\mathbb{R}) \to [0,1]$ be a credibility measure and ζ a fuzzy variable. By [14], the credibility distribution $\phi_\zeta: \mathbb{R} \to [0,1]$ associated with ζ is defined by: $\phi_\zeta(x) = Cr(\zeta \le x)$, for any $x \in \mathbb{R}$. For any $x \in \mathbb{R}$, we define by induction

$$\phi_\zeta^{[0]}(x) = \phi_\zeta(x);\ \phi_\zeta^{[k+1]}(x) = \int_{-\infty}^{x} \phi_\zeta^{[k]}(t)dt. \tag{18}$$

Definition 6.2 Let ζ and ε be two fuzzy variables and $k \in \mathbb{N}$. We define $\zeta \succcurlyeq_{cred}^{(k)} \varepsilon$ iff $\phi_\zeta^{[k]}(x) \le \phi_\varepsilon^{[k]}(x)$, for any $x \in \mathbb{R}$. If $\zeta \succcurlyeq_{cred}^{(k)} \varepsilon$, then we say that ζ *dominates credibilistically* ε *in order k*. $\succcurlyeq_{prob}^{(k)}$ is a preorder on the set of random variables $X: \Omega \to \mathbb{R}$ and $\succcurlyeq_{cred}^{(k)}$ on the set of fuzzy variables $\zeta: \mathbb{R} \to \mathbb{R}$.

Next, we study the restriction of the relation $\succcurlyeq_{cred}^{(k)}$ to the set of discrete fuzzy variables. Let ζ be a discrete fuzzy variable defined by the table:: $\zeta : \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ \mu_1 & \mu_2 & \cdots & \mu_n \end{bmatrix}$ with $a_1 < a_2 < \cdots < a_n$.

Proposition 6.3 If ϕ_ζ is the credibility distribution of ζ, then (see proof in [10]):
$\phi_\zeta(a_i) = {}^1/_2\,[max_{1\le j\le i}\mu_j + 1 - max_{1<j\le n}\mu_j]$, for any $i = 1, \dots, n$.

Proposition 6.4 The values of ϕ_ζ are given by

$$\phi_\zeta(x) = \begin{cases} 0 & if \quad x < a_1 \\ \phi_\zeta(a_1) & if \quad a_1 \le x < a_2 \\ \cdots & \\ \phi_\zeta(a_{n-1}) & if \quad a_{n-1} \le x < a_n \\ 1 & if \quad x \ge a_n \end{cases}$$

We consider now the random variable X_ζ associated with ζ: $X_\zeta : \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ p_1 & p_2 & \cdots & p_n \end{bmatrix}$ (proofs in [10]).

Proposition 6.5 The distribution function F_{X_ζ} of X_ζ coincides with ζ's credibility distribution ϕ_ζ.

Proposition 6.6 For any x∈ℝ and k ∈ ℕ, we have ${\phi_\zeta}^{[k]}(x) = {F_{X_\zeta}}^{[k]}(x)$.

Proposition 6.7 Let ζ and ε be two discrete fuzzy variables such that $\zeta \succcurlyeq_{cred}^{(k)} \varepsilon$ iff $X_\zeta \succcurlyeq_{prob}^{(k)} X_\varepsilon$, $k \in \mathbb{N}$.

7. Conclusions

In this paper risk theory was approached by credibility theory. The notion of credibilistic risk premium as an indicator of risk aversion was introduced and a relation between the credibilistic dominance of discrete fuzzy variables and the stochastic dominance of associated discrete random variables was established.

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