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A Study on Heston-Nandi GARCH Option Pricing Model

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Abstract. Heston and Nandi (2000) derive an almost closed form GARCH option pricing formula. This paper considers an implementation of the Heston and Nandi (2000)'s option pricing model. We first estimate Heston-Nandi's GARCH parameters using a time series of S&P 500 historical daily index returns from January 1981 to December 2010. Then we compare Heston and Nandi (2000)'s analytic formula with the Monte-Carlo simulation results.

Keywords: GARCH, maximum likelihood estimation, Monte-Carlo simulation.

1. Introduction

GARCH (Generalized Auto-Regressive Conditional Heteroskedasticity) models have been used to model time varying variances of asset prices and are well documented in finance literature. GARCH option pricing models are developed by Duan (1995) and Heston and Nandi (2000). Heston and Nandi (2000) derive an almost closed-form option pricing formula. Barone-Adesi, Engle, and Mancini (2008) propose a method for pricing options based on GARCH models with filtered historical non-normal innovations. Byun and Min (2010a) refine the GARCH option pricing model in Barone-Adesi, Engle, and Mancini (2008) by using the theoretical results in Christoffersen, Elkamhi, Feunou, and Jacobs (2010) and letting physical and risk-neutral one-day ahead GARCH volatilities to be different. An empirical application of Byun and Min (2010a) shows that by doing so GARCH option fitting improves significantly. Byun and Min (2010b) compare the empirical performances of several GARCH option pricing models with non-normal innovations using extensive data on S&P 500 index options.

This paper considers an implementation of the Heston and Nandi (2000)'s GARCH option pricing model. We first estimate Heston-Nandi's GARCH parameters using a time series of S&P 500 historical daily index returns from January 1981 to December 2010 (7,570 daily returns). The parameter estimates are obtained using maximum likelihood estimation (MLE) procedure. Then we compare Heston and Nandi (2000)'s analytic formula with the Monte-Carlo simulation results.

2. Heston-Nandi (2000) Model

2.1. Assumptions

The asset dynamics under the physical measure are given by

$$R_t = \ln\left(\frac{S_t}{S_{t-1}}\right) = r + \lambda h_t + \sqrt{h_t} z_t$$

S_t is the underlying asset price at time t. R_t is the log-return of the asset price. r is the continuously compounded risk-free rate. z_t is the standard normal random variable. h_t is the conditional variance of the log-return between $t-1$ and t and is known at time $t-1$. λh_t is the equity risk premium. Figure 1 shows the daily logarithmic return on the S&P 500 from January 1981 to December 2010. The dynamics of the variance is the GARCH(1,1), that is,

$$h_t = w + bh_{t-1} + a\left(z_{t-1} - c\sqrt{h_{t-1}}\right)^2$$

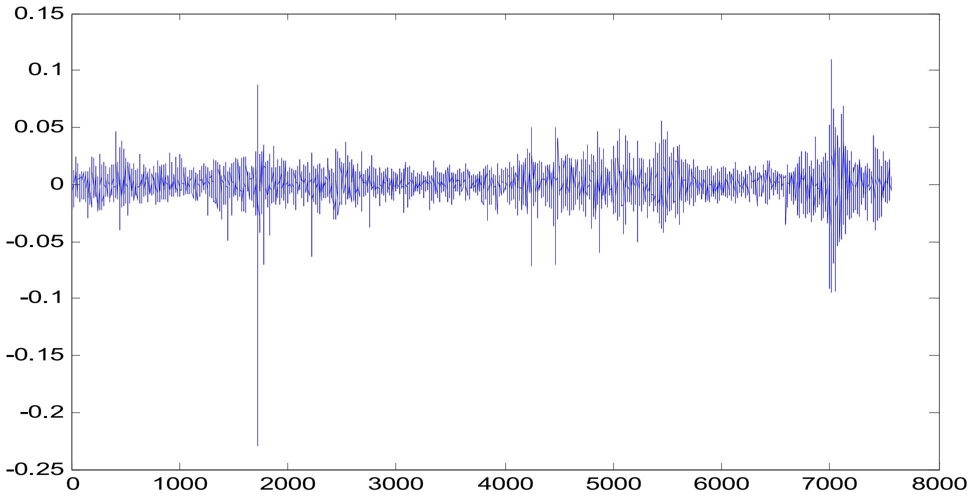

Fig. 1: Daily S&P 500 returns (1981-2010)

Heston and Nandi (2000) also assume that the value of a call option with one period to expiration obeys the Black-Scholes-Rubinstein formula. This assumption is equivalent to Duan's (1995) locally risk neutral valuation relationship (LRNVR) assumption. They use the discrete time option pricing framework of Rubinstein (1976) and Brennan (1979). Rubinstein (1976) and Brennan (1979) developed a discrete time option pricing framework by making joint conditions on the distributions as well as on the individual's preferences.

2.2. The Likelihood Function

The model has five parameters (a, b, c, w, λ). We estimate the model parameters using a time series of S&P 500 historical daily index returns from January 1981 to December 2010 (7,570 daily returns). The parameter estimates are obtained using maximum likelihood estimation (MLE) procedure. The likelihood function is

$$L = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi h_t}} \exp\left\{-\frac{(R_t - r - \lambda h_t)^2}{2h_t}\right\}$$

The log-likelihood function is

$$\log L = \sum_{t=1}^{T} -0.5\left\{\log(2\pi h_t) + \frac{(R_t - r - \lambda h_t)^2}{h_t}\right\}$$

2.3. Parameter estimates

Table I shows the maximum likelihood estimates of the Heston-Nandi model under the physical measure. We estimate the model parameters using a time series of S&P 500 historical daily index returns from January 1981 to December 2010 (7,570 daily returns). We fix the interest rate of 4% and dividend rate of 1.5%. Table I also reports the parameter estimates from Heston and Nandi (2000) and Christoffersen, Jacobs, and Ornthanalai (2009). Heston and Nandi (2000) report their parameter estimates using S&P 500 daily index returns from 1992-1994. Christoffersen, Jacobs, and Ornthanalai (2009) report their parameter estimates using S&P 500 daily index returns from 1985-2004. The last column in Table I show the parameter estimates with the constraint of $w = 0$.

Table I Maximum likelihood estimates under the physical measure

	1992-1994 HN(2000)	1985-2004 CJO(2009)	1981-2010	1981-2010 (w=0)
Parameters				
λ	0.205	2.015	1.7172	1.7686
w	5.02e-06	-9.038e-07	-1.3277e-06	0
b	0.589	0.9032	0.9138	0.8733
a	1.32e-06	4.439e-06	4.4718e-06	4.3859e-06
c	421.39	119.2	114.196	140.5724
Log-Likelihood	3,503.7	16,449	24,503	24,482

3. Option Pricing in Heston-Nandi Model

3.1. Monte-Carlo simulation

The risk-neutral process of the Heston-Nandi (2000) model can be written as

$$R_t = r - \frac{1}{2}h_t + \sqrt{h_t}z_t^*$$
$$h_t = w + bh_{t-1} + a\left(z_{t-1}^* - c^*\sqrt{h_{t-1}}\right)^2$$

z_t^* is the standard normal random variable in the risk-neutral world and $c^* = c + \lambda + 0.5$. The price at time t of a European call option with strike price K that expires at time T is given by:

$$C = e^{-r(T-t)}E_t^*[\max(S_T - K, 0)]$$

$E_t^*[\ \ ]$ is the expectation under the risk-neutral distribution. We sample paths to obtain the expected payoff in a risk neutral world and then discount this payoff at the risk-free rate.

3.2. Analytic valuation

GARCH option pricing models are developed by Duan (1995) and Heston and Nandi (2000). Duan's (1995) model is typically solved by Monte-Carlo simulation which can be slow and computationally intensive for empirical work. In contrast, Heston and Nandi (2000) derive an almost closed-form option pricing formula. They derive the following option pricing formula for the European call option with strike price K that expires at time T:

$$C = \frac{1}{2}S_t + \frac{e^{-r(T-t)}}{\pi}\int_0^\infty Re\left[\frac{K^{-i\phi}f^*(i\phi+1)}{i\phi}\right]d\phi - Ke^{-r(T-t)}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{K^{-i\phi}f^*(i\phi)}{i\phi}\right]d\phi\right)$$

In this formula, $Re[\ \ ]$ denotes the real part of a complex number. $f^*(i\phi)$ is the conditional characteristic function of the log asset price using the risk neutral probabilities. i is the imaginary number, $\sqrt{-1}$. The put option price can be obtained by put-call parity. Heston and Nandi (2000) show that the conditional generating function of the asset price under the physical measure takes the following log-linear form for the GARCH(1,1) model. This is also the moment generating function of the log asset price under the physical measure:

$$f(\phi) = E_t[S_T^\phi] = S_t^\phi \exp(A_t + B_t h_{t+1})$$

Heston and Nandi (2000) also derive the recursion formulas for the coefficients A_t and B_t. The two coefficients can be calculated recursively, by working backward from the maturity date of the option and using the terminal conditions:

$$A_t = A_{t+1} + \phi r + B_{t+1}w - \frac{1}{2}\log(1 - 2aB_{t+1})$$
$$B_t = \phi(\lambda + c) - \frac{1}{2}c^2 + bB_{t+1} + \frac{\frac{1}{2}(\phi - c)^2}{1 - 2aB_{t+1}}$$
$$A_T = B_T = 0$$

3.3. An Example

This section compares Heston and Nandi (2000)'s analytic formula and Monte-Carlo simulation results. We consider a European call option with $T = 100$ days to expiration and the following parameter values; $S = \$100,\ K = \$100,\ h(t+1) = (0.15 * 0.15)/252, r = 0.04/252, q = 0.015/252.$ The GARCH parameter values are from the last column in Table I: $\lambda = 1.7686,\ w = 0,\ b = 0.8733,\ a = 4.3859\mathrm{e}-06,$ $c = 140.5724$. The analytic formula gives the call option value of \$4.602. Figure 2 shows 95% confidence intervals for the Monte-Carlo option prices. We increase the number of simulations from 10,000 to 100,000 in Monte-Carlo simulations. The straight line in Figure 2 represents the option price from Heston and Nandi (2000)'s analytic formula.

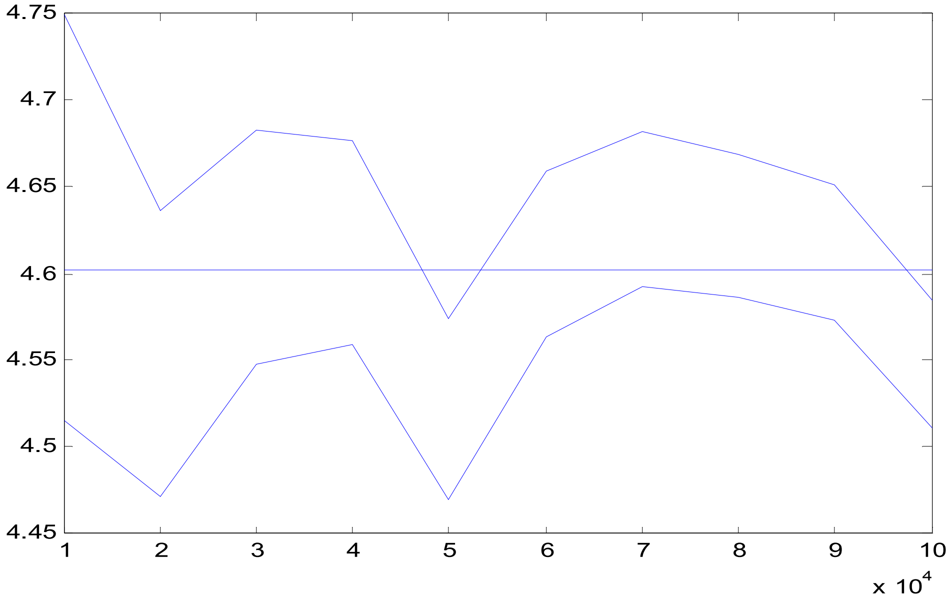

Fig. 2: 95% confidence intervals of Monte-Carlo simulations

4. Conclusion

GARCH (Generalized Auto-Regressive Conditional Heteroskedasticity) models have been used to model time varying variances of asset prices and are well documented in finance literature. Heston and Nandi (2000) derive an almost closed-form GARCH option pricing formula. This paper first estimates Heston-Nandi's GARCH parameters using a time series of S&P 500 historical daily index returns from January 1981 to December 2010 (7,570 daily returns). The parameter estimates are obtained using maximum likelihood estimation (MLE) procedure. Then we compare Heston and Nandi (2000)'s analytic formula with the Monte-Carlo simulation results.

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