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Reliability Analysis on Evaluating Process Performance with Sample Information

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Abstract. Process incapability index, C_{pp}, provide measures to determine the quality performance of a process. In fact, $C_{pp} = C_{ip} + C_{ia}$, C_{ip} denotes the potential relative expected loss, C_{ia} is the elative off-target squared. Most of the results obtained regarding the distributional and inferential properties of estimated indices were based on one single sample. In practice, however, process information is often derived from multiple samples rather than from one single sample. In this paper, we first introduce the distributional and inferential properties of the estimators of these indices based on $\overline{X}$ and S control chart samples. We then investigate the performances of the estimators of these indices based on the α-level confidence relative error for various combinations of sample size. The technique provided in this paper will be applicable when the process measurements are taken from $\overline{X}$ and S control chart.

Keywords: process control, process incapability index, non-central chi-square distribution, α-level confidence relative error.

1. Introduction

Process capability indices, including C_p, C_a, C_{pk} and C_{pm}, provide numerical measures to determine whether a process is capable of producing items within the established specification limits present by the product engineer or manufacturing engineer. Under the assumption that the process measurement X arise from a normal distribution with a mean μ and a variance σ^2, these indices are defined as (Kane, 1986; Chan et al., 1988):

$$C_p = \frac{d^*}{\sigma},\ C_a = 1 - \frac{3|\mu - T|}{d^*},\ C_{pk} = \frac{d^* - |\mu - T|/3}{\sigma} \text{ and } C_{pm} = \frac{d^*}{\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{1}$$

where LSL and USL are the lower specification limit and upper specification limit, respectively, and T denotes the target value. Besides, Greenwich and Jahr-Schaffrath (1995) introduced the process incapability index C_{pp} which transformed the index C_{pm} to provide an uncontaminated separation between information concerning the process precision and the process accuracy. The index C_{pp} is defined as:

$$C_{pp} = \left(\frac{\sigma}{d^*}\right)^2 + \left(\frac{\mu - T}{d^*}\right)^2, \tag{2}$$

where $d^* = \min\{(T - LSL)/3, (USL - T)/3\}$. If we define the first term $(\sigma/d^*)^2$ as C_{ip} (is called process imprecision index) and the second term $((\mu - T)/d^*)^2$ as C_{ia} (is called process inaccuracy index), then C_{pp} can be rewritten as $C_{pp} = C_{ip} + C_{ia}$ to provide an uncontaminated separation between information concerning process precision reflects the overall process variability C_{ip} and process accuracy reflects the departure of the process mean from the target value C_{ia}. In fact, we note that the mathematical relationships $C_{pp} = 1/(C_{pm})^2$, $C_{ia} = 9(1 - C_a)^2$ and $C_{ip} = 1/(C_p)^2$ can be established. The advantage of using C_{pp} over C_{pm} is that the estimator of the former has better

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statistical properties than that of the latter, as the former does not involve a reciprocal transformation of the process mean and variance. Lin (2006) recommended some C_{pp} values as capability requirements in most industry applications. A process is normally called inadequate if $C_{\mathrm{pp}} > 1.00$, called capable if $0.56 < C_{\mathrm{pp}} \le 1.00$, called marginally capable if $0.44 < C_{\mathrm{pp}} \le 0.56$, called satisfactory if $0.36 < C_{\mathrm{pp}} \le 0.44$, called excellent if $0.25 < C_{\mathrm{pp}} \le 0.36$, and is called super if $C_{\mathrm{pp}} \le 0.25$.

It is known that process capability indices are the functions of process mean and process standard deviation. Several quality and statistics literatures discussed the estimations of process capability indices for assessing process quality based on one single sample (Kane, 1986; Chan et al., 1988; Pearn et al., 1992; Pearn et al., 2004 and Parchami and Mashinchi, 2007). In practice, process information about process measurements is often derived from multiple samples rather than from one single sample, particularly, when a daily-based production control plan is implemented for monitoring process stability. For process information came from multiple samples, particularly, came from variable control chart samples, Li et al. (1990) gave tables of lower confidence bounds on C_{p} and C_{pk} where the sample range was substituted for the population standard deviation in the definition formula. Pearn et al. (2005) considered the problem of estimating and testing process precision based on $\bar{X}$ and R control chart and $\bar{X}$ and S control chart samples. They provided the statistical properties of the natural estimator of C_{p} and implement the hypothesis testing procedure.

Although $\bar{X}$ and R control chart is widely used in practical applications of process control (calculating sample range is easier than calculating sample standard deviation), $\bar{X}$ and S control chart is preferable to its more familiar counterparts, $\bar{X}$ and R control chart, since the range method for estimating σ loses more statistical efficiency than that standard deviation method (Montgomery, 2005). In this paper, we investigate investigate the performance of the estimator of C_{pp} index based on the α-level confidence relative error when using control chart samples. The results obtained for the accuracy of the measured process expected loss which is widely used in the manufacturing industry, relative to the control chart samples, is useful to the practitioner in determining the combination of sample size required in his application for its estimation good to the desired accuracy.

2. Estimating Process Incapability

For the case when the studied characteristic of the process is normally distributed and we have g subsamples where the sample size of the ith subsample is n. We denote this sequence of independent samples as $\{X_{i1}, X_{i2}, \ldots, X_{in}\}$, $i = 1, 2, \ldots, g$, be the characteristic value of the $N = gn$ samples with mean μ and standard deviation σ. Assume that the process is in statistical control during the time period that the subsamples are taken. Consider the process is monitored using $\bar{X}$ and S control chart. Then, for each subsample, let $\bar{X}_i = \sum_{j=1}^{g} X_{ij} / n$ and $S_i = \sqrt{\sum_{j=1}^{n} (X_{ij} - \bar{X}_i)^2 / (n-1)}$ be the ith subsample mean and the ith subsample standard deviation, respectively. $\bar{\bar{X}} = \sum_{i=1}^{g} \bar{X}_i / g$ and $\bar{R} = \sum_{i=1}^{g} R_i / g$ are the overall sample mean and the average of sample standard deviations, respectively. It notes that the mean and variance of the statistic $\bar{R}/\sigma$ are respectively given as $\mathrm{E}(\bar{R}/\sigma) = d_2$ and $\mathrm{Var}(\bar{S}/\sigma) = c_5^2/g$, where $c_4 = \sqrt{2/(n-1)}\{\Gamma(n/2)/\Gamma[(n-1)/2]\}$ and $c_5^2 = 1 - c_4^2$. Therefore, the values of c_4 and c_5^2 as determined from n.

The statistic $\bar{S}/\sigma$ is distributed approximately as $c\chi_\nu/\sqrt{\nu}$, where χ_ν is the chi distribution with ν degrees of freedom and c is constant, was most accurate (Bissell, 1990). In fact, we can derive the results of $\mathrm{E}(\bar{S}/\sigma) = \sqrt{2}c\Gamma[(\nu+1)/2]/[\sqrt{\nu}\Gamma(\nu/2)]$, $\mathrm{Var}(\bar{S}/\sigma) = (c^2/\nu)\{\nu - 2[\Gamma((\nu+1)/2)/\Gamma(\nu/2)]^2\}$. We then obtain the constant c as a function of c_4 and c_5^2, $c = \sqrt{(c_5^2/g) + c_4^2}$. Therefore, the values of c and ν are determined from g and n. Table 1 displays the corresponding c and ν for $g = 5(5)30$ and $n = 2(1)10$. In this case, Montgomery (2005) recommended estimator of σ is $\bar{S}/c_4$, ($\bar{S}/c_4$ is the unbiased estimator of σ) and the overall sample mean $\bar{\bar{X}}$ is used as an estimator for the process mean associated with $\bar{X}$ and S control chart samples. In this paper, we choice the estimator $\bar{S}/c$ to estimate σ, since $(\bar{S}/c)^2$ is the unbiased estimator of σ^2. The estimator of process incapability C_{pp} index as the following:

$$\hat{C}_{\mathrm{pp}} = \left(\frac{\nu}{N}\right)\frac{(\bar{S}/c)^2}{\mathrm{d}^{*2}} + \frac{(\bar{\bar{X}} - \mathrm{T})^2}{\mathrm{d}^{*2}}. \tag{3}$$

Table 1 The coefficients of the distribution of $\bar{S}/\sigma$ with $g = 5(5)30$ and $n = 2(1)10$

n	g					
	5		10		15	
	c	ν	c	ν	c	ν
2	0.8422	4.6251	0.8204	9.0066	0.8129	13.3869
3	0.9101	9.3961	0.8982	18.5459	0.8942	27.6952
4	0.9376	14.2819	0.9295	28.3168	0.9268	42.3512
5	0.9523	19.2256	0.9462	38.2037	0.9441	57.1815
6	0.9614	24.1739	0.9565	48.1000	0.9548	72.0259
7	0.9677	29.1568	0.9635	58.0656	0.9622	86.9742
8	0.9721	34.1357	0.9686	68.0233	0.9674	101.9107
9	0.9755	39.1367	0.9724	78.0252	0.9714	116.9137
10	0.9782	44.1327	0.9755	88.0172	0.9745	131.9015

n	g					
	20		25		30	
	c	ν	c	ν	c	ν
2	0.8092	17.7670	0.8070	22.1469	0.8055	26.5268
3	0.8922	36.8443	0.8910	45.9934	0.8902	55.1425
4	0.9254	56.3856	0.9246	70.4200	0.9240	84.4543
5	0.9431	76.1592	0.9425	95.1369	0.9421	114.1146
6	0.9540	95.9518	0.9535	119.8776	0.9532	143.8034
7	0.9615	115.8828	0.9611	144.7914	0.9608	173.7000
8	0.9668	135.7981	0.9664	169.6855	0.9662	203.5729
9	0.9709	155.8020	0.9705	194.6904	0.9703	233.5788
10	0.9741	175.7858	0.9738	219.6701	0.9736	263.5544

3. The Reliability Analysis

The α-level confidence relative error is an important and useful criterion for evaluating the reliability of the estimator. For evaluating the reliability of the estimators of process capability indices, Pearn and Lin (2002) noted that the α-level confidence relative error which is obtained from the same approach as used for finding the confidence interval, provides the practitioners with more direct and easily understood information than the confidence interval approach regarding the accuracy of their estimations and suggests a clear range on the true value of the process performance measure using the process capability index. The α-level confidence relative error of $\hat{C}_{\mathrm{pp}}$, which is defined as $\mathrm{CRE}_{\alpha}(\hat{C}_{\mathrm{pp}}) = \max_{\alpha}\{|\hat{C}_{\mathrm{pp}} - C_{\mathrm{pp}}|/C_{\mathrm{pp}}\}$. Thus, $\mathrm{CRE}_{\alpha}(\hat{C}_{\mathrm{pp}}) = e$ presents that with at least $1-\alpha$ confidence the relative deviation (relative error) of $\hat{C}_{\mathrm{pp}}$ will be no greater than e.

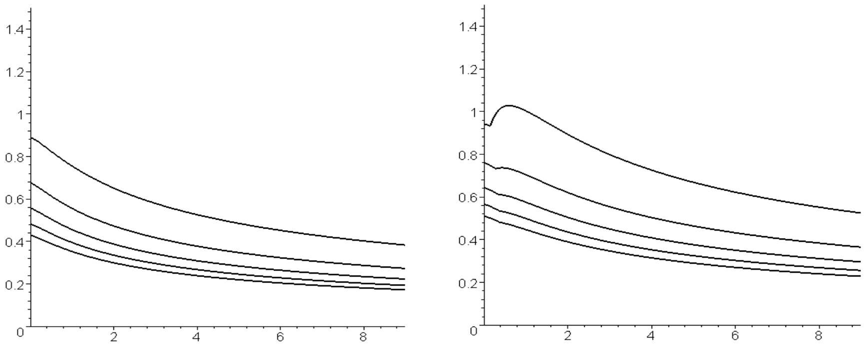
The α-level confidence relative error of the estimator of C_{pp} index can be defined as $\mathrm{CRE}_{\alpha}(\hat{C}_{\mathrm{pp}}) = \max_{\alpha}\{|\hat{C}_{\mathrm{pp}} - C_{\mathrm{pp}}|/C_{\mathrm{pp}}\} = \max_{\alpha}|(\hat{C}_{\mathrm{pp}}/C_{\mathrm{pp}}) - 1| = \max_{\alpha}\{|L_{\alpha/2} - 1|, |U_{1-\alpha/2} - 1|\}$, where $L_{\alpha/2}$ and $U_{1-\alpha/2}$ satisfy the probability equation $\Pr\{L_{\alpha/2} \le \hat{C}_{\mathrm{pp}}/C_{\mathrm{pp}} \le U_{1-\alpha/2}\} = 1-\alpha$, which can be obtained as

$$\Pr\left\{L_{\alpha/2} \le \frac{\hat{C}_{\mathrm{pp}}}{C_{\mathrm{pp}}} \le U_{1-\alpha/2}\right\} = \Pr\left\{\frac{N(C_{\mathrm{ip}} + C_{\mathrm{ia}})}{C_{\mathrm{ip}}} L_{\alpha/2} \le \zeta \le \frac{N(C_{\mathrm{ip}} + C_{\mathrm{ia}})}{C_{\mathrm{ip}}} U_{1-\alpha/2}\right\}, \tag{4}$$

where ζ is distributed as $\chi^2_{\nu+1}(\lambda)$. Therefore, the percentiles $L_{\alpha/2}$ and $U_{1-\alpha/2}$ may be obtained by finding the corresponding percentiles of the distribution. Thus,

$$L_{\alpha/2} = \frac{C_{\text{ip}} \chi^2_{\nu+1,\,\alpha/2}(\lambda)}{N(C_{\text{ip}} + C_{\text{ia}})} \text{ and } U_{1-\alpha/2} = \frac{C_{\text{ip}} \chi^2_{\nu+1,\,1-\alpha/2}(\lambda)}{N(C_{\text{ip}} + C_{\text{ia}})}, \tag{5}$$

where $\chi^2_{\nu+1,\,\alpha}(\lambda)$ is the lower αth percentile of $\chi^2_{\nu+1}(\lambda)$.



(a) $g = 5,\ \alpha = 0.05$ (b) $g = 5,\ \alpha = 0.01$

Fig. 1: The curves of $\text{CRE}_\alpha(\hat{C}_{\text{pp}})$ for (a) $g = 5,\ \alpha = 0.05$ (b) $g = 5,\ \alpha = 0.01$ with $n = 2, 4, 6, 8, 10$ (top to bottom in plot).

Table 2 $\max\{\text{CRE}_\alpha(\hat{C}_{\text{pp}})\}$ for $g = 5(5)30,\ n = 2(1)10$, and $\alpha = 0.01$

n	g					
	5	10	15	20	25	30
2	1.0276*	0.8924	0.8576	0.8324	0.8130	0.7975
3	0.8472	0.7642	0.7162	0.6839	0.6601	0.6416
4	0.7719	0.6782	0.6272	0.5938	0.5696	0.5510
5	0.7179	0.6217	0.5710	0.5382	0.5153	0.4967
6	0.6795	0.5839	0.5346	0.5030	0.4805	0.4633
7	0.6518	0.5582	0.5107	0.4804	0.4589	0.4426
8	0.6318	0.5409	0.4951	0.4662	0.4457	0.4301
9	0.6174	0.5293	0.4853	0.4576	0.4380	0.4232
10	0.6070	0.5218	0.4795	0.4530	0.4342	0.4201

* In this case, the larger value of $\text{CRE}_\alpha(\hat{C}_{\text{pp}}) = 1.0276$ is obtained at $C_{\text{ia}}/C_{\text{ip}} = 0.6081$.

We can analyze the α-level confidence relative error, $\text{CRE}_\alpha(\hat{C}_{\text{pp}})$. Since the process parameters μ and σ are unknown, then the parameter $C_{\text{ia}}/C_{\text{ip}} = (\mu - \text{T})^2/\sigma^2$ is also unknown, which has to be estimated in real applications, naturally by substituting μ and σ by $\bar{\bar{X}}$ and $\bar{S}/c$. Such an approach certainly would make this approach less reliable. To eliminate the need for further estimating the parameter $C_{\text{ia}}/C_{\text{ip}}$, we examine the behavior of the α-level confidence relative error $\text{CRE}_\alpha(\hat{C}_{\text{pp}})$ as a function of $C_{\text{ia}}/C_{\text{ip}}$. Figure 1 plots the curves of $\text{CRE}_\alpha(\hat{C}_{\text{pp}})$ versus $0 \le C_{\text{ia}}/C_{\text{ip}} \le 9$, for $g = 5$ and $n = 2(2)10$, with $\alpha = 0.05,\ 0.01$. From Figure 1, except for small values of $g,\ n$ when $\alpha = 0.01$, we find the smaller $C_{\text{ia}}/C_{\text{ip}}$, we will obtain larger value of $\text{CRE}_\alpha(\hat{C}_{\text{pp}})$, $\max\{\text{CRE}_\alpha(\hat{C}_{\text{pp}})\}$. That is, the $\text{CRE}_\alpha(\hat{C}_{\text{pp}})$ is increasing in $C_{\text{ia}}/C_{\text{ip}}$ and reaches its minimum at $C_{\text{ia}}/C_{\text{ip}} = 0$ (that is, $\mu = T$) in all cases, expect for $g = 5$ and $n = 2$ when $\alpha = 0.01$, the larger value of $\mathrm{CRE}_{\alpha}(\hat{C}_{\mathrm{pp}}) = 1.0276$ is obtained at $C_{\mathrm{ia}}/C_{\mathrm{ip}} = 0.6081$. Hence, for practical purposes we may calculate the value of $\mathrm{CRE}_{\alpha}(\hat{C}_{\mathrm{pp}})$ by setting $C_{\mathrm{ia}}/C_{\mathrm{ip}} = \hat{C}_{\mathrm{ia}}/\hat{C}_{\mathrm{ip}} = 0$ (in this case, $L_{\alpha/2} = \chi^2_{\nu+1,\alpha/2}/N$ and $U_{\alpha/2} = \chi^2_{\nu+1,1-\alpha/2}/N$) for given g, n and α, without having to further estimate the value $C_{\mathrm{ia}}/C_{\mathrm{ip}}$. Thus, based on such an approach, the decision made for sample size determination is more reliable. Table 2 displays $\max\{\mathrm{CRE}_{\alpha}(\hat{C}_{\mathrm{pp}})\}$ for $g = 5(5)30$, $n = 2(2)10$, and $\alpha = 0.01$. We find under equal total sample size, the larger *n*, we can obtain smaller value of $\mathrm{CRE}_{\alpha}(\hat{C}_{\mathrm{pp}})$. If g or n is increasing, then the value of $\mathrm{CRE}_{\alpha}(\hat{C}_{\mathrm{pp}})$ is decreasing. Table 2 displays the combination of sample size required and the corresponding minimal (conservative) reliability of the α-level confidence relative error, $\mathrm{CRE}_{\alpha}(\hat{C}_{\mathrm{pp}})$, for $\alpha = 0.01$.

4. Conclusions

Process incapability index provides measures to determine the quality performance of a process. In fact, $C_{\mathrm{pp}} = C_{\mathrm{ip}} + C_{\mathrm{ia}}$, C_{ip} denotes the process imprecision index, C_{ia} is the process inaccuracy index. In real situations where the actual values of C_{ip}, C_{ia} and C_{pp} are unknown one may estimate it by its corresponding process samples. Most of the results obtained regarding the distributional and inferential properties of estimated process capability index were based on one single sample. In practice, however, process information is often derived from multiple samples rather than from one single sample. Particularly, process measurements come from control chart samples, since the importance of using control charts first to determine if a process is in control, before estimating process capability. In this paper, we investigated the performance of the estimator of C_{pp} based on the *α*-level confidence relative error for various combinations of sample size. The results obtained for the accuracy of the estimated process incapability index which is widely used in the manufacturing industry, relative to the control chart samples, is useful to the practitioner in determining the sample size required in his application for its estimation good to the desired accuracy.

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6. References

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