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Why the determinacy condition is a weak criterion in rational expectations models.

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Abstract— **This paper disputes what Blanchard and Kahn have reported as the solution of linear rational expectation (RE) systems many years ago. Their method leads to traditional determinacy condition which is used very much nowadays. In this paper we have a new look to the mathematical procedure of this solution method and the main problem in their solution will be shown. We introduce a new methodology for modeling the systems with expectation, while in future this way of modeling can be used to replace traditional RE models.**

Keywords-Rational expectation; Determinacy condition; Stability; Uniqueness; Predictive control

I. INTRODUCTION

Linear RE models are widely used by mathematical economists in various applications especially for monetary policy analyzes. The determinacy condition is the well-known criterion, which is considered for existence of a unique stable solution. Indeed it is known as a necessary condition on the solution of RE models to be uniquely stable.

Determinacy condition was reported originally by Blanchard and Kahn (BK) [1] and then became popular among other economists [2-5]. Some of other methods which use different tools report the same condition for linear rational expectation systems as it was reported by Blanchard-Kahn [6].

However, recently Cho and McCallum [7] reported a weakness of determinacy condition. Based on a simple example, they claimed that a determinate solution may exist but differs sharply in dynamic behavior from that implied by the model considered on a sector-by-sector basis. Although they could show the weakness of determinacy condition, they did not find its reason. In their report they refer to other papers [8] to show that determinate systems might be not learnable. Also there are some other reports [9] that tried to show the weakness of determinacy condition through some examples.

In this paper, through some contradictory examples, underlying reasons the weakness of that theorem will be clarified. These examples illustrate models which hold determinacy condition but they do not have a stable solution. Indeed, the contradictory example proves that even if we accept that the determinacy condition is a necessary condition for existence of a unique stable solution, it is not sufficient.

The problems of solving RE models is not limited to Blanchard-Kahn approach [6] and so a new look on modeling of RE systems is helpful for improving their solution methods. In this paper, a new framework for modeling RE systems will be suggested which can be used for analyzing all RE systems. Then it will be shown that this framework is consistent with what had been described as RE models in the literature many years ago [11].

The paper is organized as follows. The following section reviews a summary of Blanchard-Kahn approach [1]. Next section introduces examples which show the weakness of determinacy condition. Then, we show the mathematical shortcoming of Blanchard-Kahn approach in Section IV. Section V introduces new tools to model RE systems. Finally, Section VI concludes the paper.

II. SUMMARY OF BLANCHARD-KAHN APROACH

BK considered the following canonical model [1]:

$$\begin{bmatrix} X_{t+1} \\ {}_tP_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_t \tag{1}$$

where X_0 is given initial state and

X_t: ‘predetermined variables’ (determined in t-1 or earlier)

P_t: ‘jump variables’ (choice variables determined in t)

Z_t: exogenous (random) variables $|Z_t| \le M < \infty$

${}_tP_{t+1}$: expected value of P_{t+1} at time t:

$${}_tP_{t+1} = E(P_{t+1} \mid \Omega_t) \tag{2}$$

where $E(.)$ is the mathematical expectation operator; $\Omega_{(t)}$ is the information set at t; $\Omega_t \supseteq \Omega_{t-1}$; $\Omega_{(t)}$ includes at least past and current values of X, P, Z, however it may include other exogenous variables than Z. Also, it may include future values of exogenous variables).

Let $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$ be n eigenvalues of A (counting multiplicities). Let n* be the number of eigenvalues larger than one, then:

PROPOSITION 1: If n* = dim (P_t) then there exists a unique solution.

PROPOSITION 2: If n* > dim (P_t) then there is no nonexplosive solution.

PROPOSITION 3: If n* < dim (P_t) then there are infinite solutions.

Summary of proof for PROPOSITION 1:

Solution approach relies on 'decoupeling' of forward and backward-looking model components (jumps & predetermined variables) using a diagonalization of the matrix A.

Main technical instrument is 'Jordan canoncial form':

$$A = C^{-1}JC \quad (3)$$

If A has distinct eigenvalues $\lambda_1 < \lambda_2 < .. < \lambda_n$, then the diagonal elements of J, which are the eigenvalues of A, are ordered by increasing absolute value.,

$$J = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \quad (4)$$

Now we have,

$$\begin{bmatrix} X_{t+1} \\ {}_tP_{t+1} \end{bmatrix} = C^{-1}JC \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_t$$
$$\Rightarrow C \begin{bmatrix} X_{t+1} \\ {}_tP_{t+1} \end{bmatrix} = JC \begin{bmatrix} X_t \\ P_t \end{bmatrix} + C\gamma Z_t \quad (5)$$

J, C, C^{-1}, and γ are decomposed accordingly,

$$\mathrm{J} = \begin{bmatrix} \mathrm{J}_1 & 0 \\ 0 & \mathrm{J}_2 \end{bmatrix}, C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$
$$C^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \text{ and } \gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \quad (6)$$

where J is partitioned so that all eigenvalues of J_1 are on or inside the unit circle, all eigenvalues of J_2 are outside the unit circle.

Consider the transformation

$$\begin{bmatrix} Y_t \\ Q_t \end{bmatrix} = \mathrm{C} \begin{bmatrix} X_t \\ P_t \end{bmatrix} \quad (7)$$

Premultiplying both sides of system equation by C, and using $A = C^{-1}JC$,

$$\begin{bmatrix} {}_tY_{t+i+1} \\ {}_tQ_{t+i+1} \end{bmatrix} = \begin{bmatrix} \mathrm{J}_1 & 0 \\ 0 & \mathrm{J}_2 \end{bmatrix} \begin{bmatrix} {}_tY_{t+i} \\ {}_tQ_{t+i} \end{bmatrix} + C\gamma_t Z_{t+i} \quad (8)$$

As C is invertible, knowledge of X, and P, is equivalent to knowledge of Y_t and Q_t: the transformation does not affect $\Omega_{(t)}$. Also existence (uniqueness) of a solution in this system is equivalent to existence (uniqueness) of a solution of the main system.

Equation (8) is composed of two subsystems. The equation of the first subsystem is

$${}_tY_{t+i+1} = \mathrm{J}_1 {}_tY_{t+i} + \left(C_{11}\gamma_1 + C_{12}\gamma_2\right)_t Z_{t+i} \quad \forall i \geq 0. \quad (9)$$

By construction of J_1 this system is stable or borderline stable. However by construction of J_2, the second subsystem might explode and violate the nonexplosion condition, unless

$$Q_t = -\sum_{i=0}^{\infty} J_2^{-i-1} \left(C_{21}\gamma_1 + C_{22}\gamma_2\right)_t Z_{t+i} \quad (10)$$

uniquely determines Q_t. Thus existence and uniqueness of the solution of the main system depends on existence and uniqueness of the sequence of Q_t and Y_t. Y_t just has to satisfy (9).

III. EXAMPLES

EXAMPLE 1: As our basic example consider the following linear model

$$\begin{bmatrix} X_{t+1} \\ {}_tP_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_t$$
$$with, \quad A = \begin{bmatrix} 1.4 & 0 \\ 0.8 & 0.6 \end{bmatrix} and\ \gamma = \begin{bmatrix} 1.1 \\ 0.2 \end{bmatrix} \quad (11)$$

Determinacy condition implies that the system has a unique stable solution if and only if A has one unstable eigenvalue. Based on BK method [1], since there are one unstable eigenvalue, the system is determinated and therefore there should be a unique stable solution for this model.

The equation of this example is composed of two subsystems. The equation of the first subsystem is

$$X_{t+1} = 1.4X_t + 1.1Z_t \quad (12)$$

The relevant eigenvalue of this system is *1.4* and so the behavior of X_t is explosive.

We conclude that in this example the model holds the determinacy condition but it does not have a stable solution.

EXAMPLE 2: Suppose a model

$$X_{t+1} = aX_t + b\,Z_t + n_{t+1} \quad (13)$$

where a and b are constant values and $n_{(t+1)} \sim iid(0,6)$.

To have a stable solution for this system the absolute value of a must be smaller than one.

Taking expectation from both sides of the above equation and using the law of iterated expectation[1] we have:

$${}_tX_{t+1} = a\,{}_tX_t + b\,{}_tZ_t + {}_t\,n_{t+1} = aX_t + b\,Z_t \quad (14)$$

This equation is equivalent to the previous equation but determinacy condition implies that this new equation has a unique stable solution if and only if the absolute value of a is greater than one.

One might argue that in this example $X_{(t)}$ is not a nonpredetermined variable. But this example is only for showing that the same forms of equations are used for different variables in literature.

We should note that if an eigenvalue is stable in forward-looking solution it would be unstable in the form of backward-looking solution.

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[1] We summarize this law from10. Blanchard, O. and S. Fischer, *Lectures on macroeconomics*. 1989: The MIT press.].

The law of iterated expectations: let Ω be an information set and ω be a subset of this information set. Then for any variable x,

$E[E[x \mid \Omega] \mid \omega] = E[x \mid \omega]$

Or, heuristically, for rational expectation systems the law of iterated expectation is rewritten as follow,

$E[E[x \mid I_{t+1}] \mid I_t] = E[x \mid I_t]$

EXAMPLE 3: Consider the following system equation which is applied and analyzed frequently for different purposes in [10],

$$y_{(t)} = a_t y_{t+1} + cx_t \quad (15)$$

where *a, c* are constant values. The recursive solution of this system which is found in [10] is as follow,

$$y_t = c\sum_{i=0}^{T} a^i {}_t x_{t+i} + a^{T+1} {}_t y_{t+T+1} \quad (16)$$

It is said that if |a|<1 the following solution can be found for the above system,

$$y_t = c\sum_{i=0}^{\infty} a^i {}_t x_{(t+i)} \quad (17)$$

We should note that finding the solution of RE systems is composed of two steps. In the first step the best estimation for expectation term will be found and in the second step the convergence condition will be studied. Although these two steps are not separated in the literature, it is known that without two aforementioned steps the solution cannot be found.

To find the estimation of the system expectation term we use following equation,

$${}_t y_{t+1} = \frac{1}{a} y_t + \frac{c}{a} x_t \quad (18)$$

It is now clear that if the estimation of the expectation term is not to explode we should have $|\frac{1}{a}|<1$ or equivalently $|a|>1$. Usually the first step is ignored and there is no argument about the estimation of the expectation term and so they assume that having $|a|<1$ is satisfactory for finding the solution. In this simple example we can conclude that having an eigenvalue with absolute value greater than one is not the necessary and sufficient condition for stability of the whole system. Nevertheless, if $|a|<1$ then the unique bounded solution of (16) is as stated in (18) if the system satisfies $a^{T+1} {}_t y_{t+T+1} = 0$.

IV. THE ORIGIN OF THE MISTAKE

Let us rewrite the equation of Q_t in BK as follows:

$${}_t Q_{t+i+1} = J_2 \, {}_t Q_{t+i} + \left(C_{21}\gamma_1 + C_{22}\gamma_2\right)_t Z_{t+i} \quad (19)$$

The recursive solution of the above equation is,

$$Q_t = J_2^{-q-1} {}_t Q_{t+q+1} - \sum_{i=0}^{q} J_2^{-i-1}\left(C_{21}\gamma_1 + C_{22}\gamma_2\right)_t Z_{t+i} \quad (20)$$

So the recursive solution of this subsystem is equivalent to (11) if and only if

$$\lim_{q\to\infty} J_2^{-q-1} {}_t Q_{t+q+1} = 0 \quad (21)$$

Although $\lim_{q\to\infty} J_2^{-q-1} = 0$, remember that $Q_{(t+q+1)}$ can explode as it can be inferred from (20) that we have,

$${}_t Q_{(t+q+1)} = J_2^{q+1}(Q_t - \sum_{i=0}^{q} J_2^{-i-1}\left(C_{21}\gamma_1 + C_{22}\gamma_2\right)_t Z_{t+i}) \quad (22)$$

Therefore, in a general case we have to analyze if the result is acceptable or not.

In a special case where ${}_t Z_{t+1} = 0$, the system equation can be rewritten in the following form,

$$Q_t = J_2^{-q-1} {}_t Q_{t+q+1} \; or \; {}_t Q_{t+q+1} = J_2^{+q+1} Q_t \quad (23)$$

It is clear that the result is different from BK approach in (10) unless $Q_t = {}_t Q_{t+q+1} = 0$.

Suppose that this new condition $(Q_t = {}_t Q_{t+q+1} = 0)$ is satisfied then it can be rewritten as,

$$\begin{aligned} X_{t+i} &= B_{11} {}_t Y_{t+i} + B_{12} {}_t Q_{t+i} = B_{11} {}_t Y_{t+i} \\ {}_t P_{t+i} &= B_{21} {}_t Y_{t+i} + B_{22} {}_t Q_{t+i} = B_{21} {}_t Y_{t+i} \end{aligned} \quad (24)$$

BK assumed that B_{11} is invertible so in this case it is written,

$${}_t P_{t+i} = B_{21} B_{11}^{-1} X_{t+i} \quad (25)$$

It can be seen that to satisfy BK condition there should be an algebraic relationship between *X* and *P* in this case. Because of this algebraic relationship the dynamics of these two variables should be similar and so it is not possible to say that *X* is predetermined and *P* is nonpredetermined variable.

So for arbitrary values of *Z* if the BK condition should be satisfied it would impose some algebraic constrains which are not satisfied in general cases.

To summarize this section we can say that BK implicitly assumes that (21) is satisfied in all cases; in other words they replaced the value of $\lim_{q\to\infty} J_2^{-q-1} {}_t Q_{t+q+1}$ by zero. However, this term does not tend to zero for all Z_t. Indeed, since ${}_t Q_{t+q+1}$ is dependent of J_2, the limit is indeterminate and its evaluation imposes extra conditions that might not be satisfied in all cases.

Although, if the extra conditions are satisfied the BK solution can be used for the system but in general cases it is not satisfied. We conclude that adding unstable eigenvalues imposes new conditions which are not satisfied in all cases and so this method should be revised.

V. NEW WAYS OF ANALYZE

A. *Model overview*

In previous parts, the weakness of some classical approaches for analyzing RE models have been shown. In this part a new method which can be used in future works will be suggested.

The way that expectation is made, plays the main role in modeling of RE systems. This point was noted in main article of RE systems by Muth[11] but unfortunately it is now common in literature to write the equations without concerning about the origin of the expectation terms. In other word we can say that Muth has suggested such a structure but it hasn't been modeled in a correct way yet.

An economic system includes various agents. These agents estimate the system based on their information and behave due to their expectation of future variables to minimize their own cost function.

Fig.1 shows the simplest form we can model such behavior. In this figure we have two different agents with an estimator and a controller. They predict the future behavior of the variables and behave in a way that, based on their estimation, has the best trajectory for their own welfare function.

Estimation of each agent is based on the past values of other parts of economic system, the past value of its own output and the known trajectory of future inputs to the whole system.

The expectation of future variables is found in the estimator part and then it is used in the predictive controller. So the output of the predictive controller includes the effect of agents' prediction. Writing system equations based on each agent viewpoint tends to a similar equation form as in RE system.

We can say that in actual economic systems we have such structure but it is common not to separate each part and model them as a unique system with some dynamic equations. But to analyze the system it is more useful to separate them.

This structure can be called multiagent predictive control which has a common framework in control engineering. The stability, learnability, performance analysis and robustness analysis of such systems are widely analyzed in engineering and they can be used in economic systems.

The behavior of this structure was described by Muth[11] many years ago but it hasn't been used in this form yet.

B. *A simple model*

Suppose that we have the following system,

$$X_{t+1} = AX_t + BU_t + CD_{t+1} \qquad (26)$$

with

$$X_{t+1} = \begin{bmatrix} X_{1(t+1)} \\ X_{2(t+1)} \end{bmatrix}, D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \quad U_t = \begin{bmatrix} U_{1(t)} \\ U_{2(t)} \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad (27)$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

where,

X_t: State variables of other parts of economic system

U_t: Exogenous variable for states of other parts

D_t: An iid noise

A, B,C: Constant matrices of appropriate size.

Note that $U_{1(t)}$ is a exogenous variable in view point of the second agent and vice versa.

Suppose that the first agent has no estimation of the structure which is used by second agent to form its controller. So the best estimation of this system based on current information from the first agent view point is,

$${}_tX_{t+1} = AX_t + BU_t \qquad (28)$$

Finding the expectation for q steps ahead results,

$${}_tX_{t+q} = (A)^q X_t + \sum_{i=1}^{q} (A)^{q-i} BU_{t+i-1} \qquad (29)$$

Each agent tries to minimize its Loss function which is in the following form,

$$L_{1(t)} = \sum_{w=1}^{n} \beta^{w-1} ({}_tX_{t+w} - X_{d,t+w})({}_tX_{t+w} - X_{d,t+w})^T \qquad (30)$$

where, $\beta \in (0,1)$ is the discount factor, $X_{d,t+w}$ is the desired value of the states for this agent and and n is the optimization horizon. This agent might use the following form of decision making (controller) to minimize its own Loss function,

$$U_{1(t)} = f\left(X_t, U_2(t+k), X_{d(t+k)}\right) \qquad (31)$$

where f is a function which is found by minimization procedure. Knowing this structure the system equation is rewritten as,

$$X_{t+1} = AX_t + b_2U_{2(t)} + CD_{t+1} + b_1f\left(X_t, U_2(t+k), X_{d(t+k)}\right) \qquad (32)$$

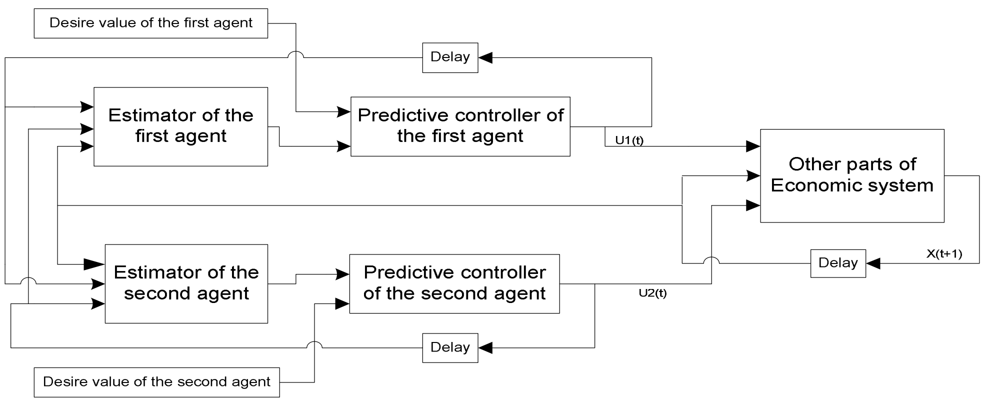

Figure 1. An structure for economic systems with rational expectation dynamic

The above equation can be used for describing the system equation from the second agent point of view. This equation is in the form the solutions of RE systems.

The main difference between the solution of this model and classical solution of RE models is modeling the expectation terms in the Loss function of each agent instead of in their dynamic. Therefore, it can be concluded that in an RE model the same structure exists.

In a special case if the optimization horizon in the Loss function for the first agent is set to be *n=3* then *f* can be found by a routine procedure[2].

It should be noted that estimation of states at time *t+d* depends on choosing future input signals and so it is necessary to make some assumptions about them. One possibility is to assume that the first agent has some information about input signals which are implied by the second agent and its own input signal will remain constant for the optimization horizon. In other word we assume that $U_{1(t)} = U_{1(t+2)} = U_{1(t+3)}$ and $_t U_{2(t+k)}$ $k = 1,2,...$ are known.

Another simplification is needed to have a simple stability analysis for the above system. In this simple structure we assumed that the second agent is not using a feedback rule. Although in a real world most of the agents use feedback rules, in this simplified example we reduce the complexity of the real world to show the main concept.

With the above assumptions and replacing the optimal solution for the first agent the stability of the system is analyzed by the eigenvalues of the following closed-loop transition matrix,

$$A_{cl} = A - F^{-1}G \quad (33)$$

with

$$F = \begin{pmatrix} \beta^2 \left(A^2 b_1 + A b_1 + b_1\right)^T \left(A^2 b_1 + A b_1 + b_1\right) + \\ +\beta\left(A b_1 + b_1\right)^T \left(A b_1 + b_1\right) + (b_1)^T b_1 \end{pmatrix}$$
$$G = \begin{pmatrix} +\beta\left(A b_1 + b_1\right)^T A^2 + b_1^T A + \\ +\beta^2 \left(A^2 b_1 + A b_1 + b_1\right)^T A^2 \end{pmatrix} \quad (34)$$

C. A simple example

Suppose that the system structure is in the same form of (26) and its variables are as follow,

$$A = \begin{bmatrix} 1.3 & 0.7 \\ 0.2 & 0.1 \end{bmatrix}, B = \begin{bmatrix} 2.4 & 0.6 & 1 \\ 1.3 & 2 & 1.4 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$D_1 \sim iid(0,0.05), D_2 \sim iid(0,0.01), \quad (35)$$
$$U_{2(t)} = 2 - e^{-t/n}, X_{d(t)} = \begin{bmatrix} 0.5 - e^{-3t/N} \\ 1.4 + e^{-2t/N} \end{bmatrix}$$

where N is the simulation horizon. The discount factor is set to be β=0.8.

The eigenvalues of the closed-loop are $\begin{bmatrix} 0.8671 & -0.2667 \end{bmatrix}$ and so the closed loop system is stable.

VI. CONCLUSION

Based on various simple examples, determinacy condition was shown to be not really acceptable in some practical cases and so there is a need for its revision.

Multiagent predictive control approaches was suggested to be used for modeling RE systems, while this approach was new in economic literature, though it is consistence with economic description of RE systems [11]. Easiness of this approach for analyzing the stability and performances of various systems were demonstrated which proves the drastic potential of this methodology as a strong tool in the literature.

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[2] The equations of this procedure can be sent by email on request.